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* Introduction to the Mechanics of Continua

So far we have treated only point-like particles; an exception was the treatment of the rigid body in Chap. 8. In this chapter we will investigate deformable bodies, for a twofold reason:

(1) If a body is not rigid but deformable, it has its own internal dynamics, which is independent of its state of motion (disregarding inertial forces), as was investigated, e.g., in Chap. 8.

(2) The continuum mechanics is a first example of a field theory. A more detailed treatment of a field theory is the subject of the Maxwell theory of the electromagnetic fields.

Comment: Here, we have elastic (and not plastic) deformations in mind. The former apply to those systems which return to their original state after the external forces are released, while the latter remain in a deformed state.

10.1 Fields

If one wants to treat for example the mechanics of a vibrating violin string, then one is not interested in the atomistic character of the string, but rather in its mass density, which, in combination with the stress, determines the pitch, i.e., the frequency. Differently from Chap. 8 one deals here not with a rigid, but with a deformable body.

In the mechanics of point masses (index $i$) one investigates the time dependence of the coordinates $r_i(t)$; there is but one variable: the time $t$. The dynamics of a mass element $dm$ at the position $r$ of the continuum is described by the time dependence of each of the three components of the displacement field $u(r, t)$; here the variables are the time $t$ and the position $r$. One must thus

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1 Goldstein [1], Chap. 12 (2nd ed.); in most of the other text books on Classical Mechanics the mechanics of the continua is omitted.

2 See the Course on Electromagnetism.
distinguish: In point mechanics $r$ is a coordinate, and in continuum mechanics $r$ is a variable. The velocity is $\dot{r}$ in point mechanics, and the velocity field is $\mathbf{u}(r, t)$, etc. The generalized coordinates are $q(t)$ and $q(r, t)$, respectively.

**Densities**

Essential properties of the continua are (position and possibly time dependent) densities, like the mass density, momentum density, energy density, etc.

The dynamics of fields can be derived alternatively from the Newtonian equations or from a variational principle. In the latter case the Lagrangian and Hamiltonian densities are the starting point of a (classical and possibly also nonclassical) field theory. The theory leads to equations of motion for the fields (thus to the displacement field in the example of the string).

The transition from (atomistic) point masses to the continuous mass density is made as in Sect. 8.3.

**Agreement**

- For reasons of simplification the four variables $t = x^0$ and $x^i$ ($i = 1, 2, 3$) shall be combined into a vector-like (contravariant) quantity $x^\mu$ with an upper Greek index $\mu$ (similar to what is done in Special Relativity$^3$),

$$
(t, r) = x^\mu \quad \mu = 0, 1, 2, 3.
$$

Similarly the space derivatives

$$
\frac{\partial}{\partial x^i} \equiv \partial_i \quad i = 1, 2, 3
$$

and the time derivative

$$
\frac{\partial}{\partial t} \equiv \partial_0
$$

shall be combined in the (covariant) quantity $\partial_\mu$ with lower index,

$$
\partial_\mu \equiv \left( \frac{\partial}{\partial t}, \nabla \right).
$$

- The derivative with respect to $x^\mu$ is also occasionally denoted by an index with a preceding comma,

$$
\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f \equiv f_{,\mu} \quad \partial_\mu q_l \equiv q_{l,\mu},
$$

etc.

$^3$ In Relativity Theory one defines $x^0 = ct$. 