The Goldblatt-Thomason Theorem for Coalgebras

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Abstract. Goldblatt and Thomason’s theorem on modally definable classes of Kripke frames and Venema’s theorem on modally definable classes of Kripke models are generalised to coalgebras.

1 Introduction

The Goldblatt-Thomason theorem [11] states that a class of Kripke frames closed under ultrafilter extensions is modally definable if and only if it reflects ultrafilter extensions and is closed under generated subframes, homomorphic images and disjoint unions. The proof is based on the duality between Boolean algebras and sets

\begin{equation}
\begin{array}{ccc}
\mathcal{BA} & \overset{\Pi}{\longrightarrow} & \mathcal{Set}^{\mathcal{OP}} \\
\overset{\Sigma}{\longleftarrow} & \ & \ \end{array}
\end{equation}

where $\Pi$ is powerset and $\Sigma$ assigns to a $\mathcal{BA}$ the set of ultrafilters. $\Sigma$ is left-adjoint to $\Pi$ but, of course, this adjunction does not form a dual equivalence. The price we have to pay for this is that going from $\mathcal{Set}$ to $\mathcal{BA}$ and back leaves us with $\Sigma \Pi X$: If $X$ is the carrier of a Kripke frame, then its ultrafilter extension has carrier $\Sigma \Pi X$, which explains why ultrafilter extensions appear in the theorem.

Our generalisation from Kripke frames to $T$-coalgebras works as follows. $\mathcal{Set}$ and $\mathcal{BA}$ are completions (with filtered colimits) of the categories $\mathcal{Set}_\omega$ of finite sets and $\mathcal{BA}_\omega$ of finite Boolean algebras, respectively. $\mathcal{BA}_\omega$ and $\mathcal{Set}_\omega$ are dually equivalent. Now, given a functor $T$ on $\mathcal{Set}$ that preserves finite sets, we can restrict $T$ to $\mathcal{Set}_\omega$. Via the dual equivalence $\mathcal{BA}_\omega \simeq \mathcal{Set}_\omega^{\mathcal{OP}}$, this gives us a functor on $\mathcal{BA}_\omega$, which we can then lift to a functor $L : \mathcal{BA} \to \mathcal{BA}$.

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\end{equation}

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showed the following: (i) \( L \) has a presentation and therefore determines a logic for \( T \)-coalgebras, (ii) \( \Pi \) extends to a functor \( \text{Coalg}(T) \to \text{Alg}(L) \), (iii) if \( T \) weakly preserves cofiltered limits, then \( \Sigma \) extends to a map on objects \( \text{Alg}(L) \to \text{Coalg}(T) \). This note shows that the classical Goldblatt-Thomason theorem generalises to those \( T \)-coalgebras where \( \Sigma : \text{BA} \to \text{Set} \) can be extended to a functor \( \text{Alg}(L) \to \text{Coalg}(T) \).

The same argument also generalises a similar definability result for Kripke models due to Venema [22].

**Related Work.** An algebraic semantics for logics for coalgebras and its investigation via the adjunction between \( \text{BA} \) and \( \text{Set} \) has been given in Jacobs [13]. The idea that a logic for \( T \)-coalgebras is a functor \( L \) on \( \text{BA} \) appears in [5,15] and can be traced back to Abramsky [12] and Ghilardi [10]. It has been further developed in [6,16]. The general picture underlying diagram (2) has been discussed in Lawvere [19] where it is attributed to Isbell. The implications of this Isbell-conjugacy for logics for coalgebras are explained in [17]. For topological spaces, which can be seen as particular coalgebras, the Goldblatt-Thomason theorem is due to Gabelaia [9] and ten Cate et al [7].

## 2 Coalgebras and Their Logics

**Definition 2.1.** The category \( \text{Coalg}(T) \) of coalgebras for a functor \( T \) on a category \( \mathcal{X} \) has as objects arrows \( \xi : X \to TX \) in \( \mathcal{X} \) and morphisms \( f : (X, \xi) \to (X', \xi') \) are arrows \( f : X \to X' \) such that \( Tf \circ \xi = \xi' \circ f \).

Examples of functors of interest to us in this paper are described by

**Definition 2.2 (gKPF).** A generalised Kripke polynomial functor (gKPF) \( T : \text{Set} \to \text{Set} \) is built according to

\[
T ::= \text{Id} \mid K_C \mid T + T \mid T \times T \mid T \circ T \mid \mathcal{P} \mid \mathcal{H}
\]

where \( \text{Id} \) is the identity functor, \( K_C \) is the constant functor that maps all sets to a finite set \( C \), \( \mathcal{P} \) is covariant powerset and \( \mathcal{H} \) is \( 2^2 \).

**Remark 2.3.** The term ‘Kripke polynomial functor’ was coined in Rößiger [20]. We add the functor \( \mathcal{H} \). \( \mathcal{H} \)-coalgebras are known as neighbourhood frames in modal logic and are investigated, from a coalgebraic point of view, in Hansen and Kupke [12].

We describe logics for coalgebras by functors \( L \) on the category \( \text{BA} \) of Boolean algebras. Although this approach differs conceptually from Jacobs’s [13], the equations appearing in the example below are the same as his.