Packing and Covering $\delta$-Hyperbolic Spaces by Balls*

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Abstract. We consider the problem of covering and packing subsets of $\delta$-hyperbolic metric spaces and graphs by balls. These spaces, defined via a combinatorial Gromov condition, have recently become of interest in several domains of computer science. Specifically, given a subset $S$ of a $\delta$-hyperbolic graph $G$ and a positive number $R$, let $\gamma(S, R)$ be the minimum number of balls of radius $R$ covering $S$. It is known that computing $\gamma(S, R)$ or approximating this number within a constant factor is hard even for 2-hyperbolic graphs. In this paper, using a primal-dual approach, we show how to construct in polynomial time a covering of $S$ with at most $\gamma(S, R)$ balls of (slightly larger) radius $R + \delta$. This result is established in the general framework of $\delta$-hyperbolic geodesic metric spaces and is extended to some other set families derived from balls. The covering algorithm is used to design better approximation algorithms for the augmentation problem with diameter constraints and for the $k$-center problem in $\delta$-hyperbolic graphs.

Keywords: covering, packing, ball, metric space, approximation algorithm.

1 Introduction

The set cover problem is a classical question in computer science [39] and combinatorics [9]. In this problem, given a collection $S$ of subsets of a domain $U$ with $n$ elements, the task is to find a subcollection of $S$ of minimum size $\gamma(S)$ whose union is $U$. It was one of Karp’s 21 NP-complete problems. More recently, it has been shown that, under the assumption $P \neq NP$, set cover cannot be approximated in polynomial time to within a factor of $c \cdot \ln n$, where $c$ is a small constant; see [3] and the references cited therein. On the other hand, set cover can be approximated in polynomial time to within a factor of $\ln n + 1$ using several algorithms [39], in particular, using the greedy algorithm. The set packing problem asks to find a maximum number $\pi(S)$ of pairwise disjoint subsets of $S$. Another problem closely related to set cover is the hitting set problem. A subset

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T is called a hitting set of $S$ if $T \cap S \neq \emptyset$ for any $S \in S$. The minimum hitting set problem asks to find a hitting set of $S$ of smallest cardinality $\tau(S)$.

Numerous algorithmic and optimization problems can be formulated as set cover or set packing problems for structured set families. For example, many papers consider cover and packing problems with set families like intervals and unions of intervals of a line, subtrees of a tree, or cliques, cuts, paths, and balls of a graph. For example, in case of covering with balls, one can expect that the specific metric properties of graphs in question yield better algorithmic results in comparison with the general set cover. Although the set cover problem can be viewed as a particular instance of covering with unit balls of rather special graphs, for several graphs classes polynomial time algorithms have been designed. These algorithms resides on the treelike structure of those graphs and on the equality between ball covering and packing numbers of such graphs.

In this note, we consider the problem of covering and packing by balls and union of balls of hyperbolic metric spaces and graphs. The ball $B(x, R)$ of center $x$ and radius $R \geq 0$ consists of all points of a metric space $(X, d)$ at distance at most $R$ from $x$. In our paper, we will consider covering and packing problems of the following type: given a finite subset $S$ of points of $X$, a radius $R$, and a slack parameter $\delta$, find a good covering of $S$ with balls of radius at most $R + \delta$. We show that if the metric space $(X, d)$ is $\delta$-hyperbolic, then in polynomial time we can construct a covering of $S$ with balls of radius $R + \delta$ and a set of the same size of pairwise disjoint balls of radius $R$ centered at points of $S$. This type of results is obtained for arbitrary subfamilies of balls and for set-families consisting of unions of $\kappa$ balls of $(X, d)$. We apply these results to design better approximation algorithms for the $k$-center problem and the augmentation problem with diameter constraints in $\delta$-hyperbolic graphs.

1.1 Geodesic and $\delta$-Hyperbolic Metric Spaces

Let $(X, d)$ be a metric space. A geodesic segment joining two points $x$ and $y$ from $X$ is a map $\rho$ from the segment $[a, b]$ of length $|a - b| = d(x, y)$ to $X$ such that $\rho(a) = x, \rho(b) = y$, and $d(\rho(s), \rho(t)) = |s - t|$ for all $s, t \in [a, b]$. A metric space $(X, d)$ is geodesic if every pair of points in $X$ can be joined by a geodesic. We will denote by $[x, y]$ any geodesic segment connecting the points $x$ and $y$. Every graph $G = (V, E)$ equipped with its standard distance $d_G$ can be transformed into a (network-like) geodesic space $(X, d)$ by replacing every edge $e = (u, v)$ by a segment $[u, v]$ of length 1. These segments may intersect only at their commons ends. Then $(V, d_G)$ is isometrically embedded in a natural way in $(X, d)$.

Introduced by Gromov [29], $\delta$-hyperbolicity measures, to some extent, the deviation of a metric from a tree metric. Recall that a metric space $(X, d)$ embeds into a tree network (with positive real edge lengths), that is, $d$ is a tree metric, if and only if for any four points $u, v, w, x$ the two larger ones of the distance sums $d(u, v) + d(w, x), d(u, w) + d(v, x), d(u, x) + d(v, w)$ are equal. Now, a metric space $(X, d)$ is called $\delta$-hyperbolic if the two larger distance sums differ by at most $\delta$. A connected graph $G = (V, E)$ equipped with standard graph metric $d_G$ is $\delta$-hyperbolic if $(V, d_G)$ is a $\delta$-hyperbolic metric space.