17 Squeezed States of Light

The Heisenberg uncertainty principle $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$, between the standard deviations of two arbitrary observables, $\Delta A = \langle (A - \langle A \rangle)^2 \rangle^{1/2}$ and similarly for $\Delta B$, has a built-in degree of freedom: one can squeeze the standard deviation of one observable provided one “stretches” that for the conjugate observable. For example, the position and momentum standard deviations obey the uncertainty relation

$$\Delta x \Delta p \geq \hbar / 2 , \quad (17.1)$$

and we can squeeze $\Delta x$ to an arbitrarily small value at the expense of accordingly increasing the standard deviation $\Delta p$. All quantum mechanics requires is than the product be bounded from below. As discussed in Sect. 13.1, the electric and magnetic fields form a pair of observables analogous to the position and momentum of a simple harmonic oscillator. Accordingly, they obey a similar uncertainty relation

$$\Delta E \Delta B \geq (\text{constant}) \hbar / 2 . \quad (17.2)$$

In principle, we can squeeze $\Delta E$ at the expense of stretching $\Delta B$, or vice versa. Such squeezing of the electromagnetic field has recently been the object of considerable attention in view of potential applications to high precision measurements. It offers the promise of achieving quantum noise reduction beyond the “standard shot noise limit” and might find applications in phase-sensitive detection schemes as required for the detection of gravitational radiation [see Meystre and Scully (1983)].

As an electromagnetic field mode oscillates, energy is transferred between $E$ and $B$ each quarter period. Hence, to observe the squeezing in $\Delta E$, we must somehow select its active quadratures from the general electromagnetic oscillation. In effect we need an optical frequency “chopper” that spins around opening up to show us a single quadrature. By varying the relative phase of the chopper and the light, we can look at any quadrature. In practice heterodyne detection provides such a selection process. It effectively multiplies the signal to be measured by a sine wave whose peaks correspond to openings in the chopper. By varying the relative phase of the signal and heterodyne waves, we can examine any quadrature of the signal. Generating the right single-mode radiation complete with a suitable heterodyne wave seems to
be very difficult. Instead, we can use four-wave mixing processes described in Sect. 2.4 and in Chap. 10 to create a pair of waves whose sum exhibits squeezing. The heterodyne wave is then derived directly from the pump wave.

Section 17.1 describes how squeezing in one quadrature results in stretching in the orthogonal quadrature for the simple harmonic oscillator. This treatment is a squeezed-state generalization of the Schrödinger oscillating wavepacket that forms the basis for coherent states of light. This section also introduces the squeezing operator and shows how it turns the circular variance of the coherent state in the complex $a$ plane into an ellipse. Section 17.2 extends the single-sidemode master equation of Sect. 16.4 to treat the two-side-mode cases found in three- and four-wave mixing. This theory quantizes the signal and conjugate waves, while leaving the pump wave classical. Section 17.3 applies this formalism to calculate the variances for squeezing via multiwave mixing. It gives some numerical illustrations for two-level media. Section 17.4 develops the theory of a squeezed “Vacuum” and shows how an atom placed in such a vacuum has two dipole dephasing rates, a small one in the squeezed quadrature, and a correspondingly larger one in the stretched quadrature.

### 17.1 Squeezing the Coherent State

Section 13.4 shows how a displaced ground state of the simple harmonic oscillator of the correct width oscillates back and forth with unchanging width. However, if we now squeeze this wavepacket, it will spread for a quarter of a cycle, then return to the squeezed value at the half cycle, and so on as illustrated in Fig. 17.1. Looking at the mean and standard deviation of the electric field vector in the complex $a$ plane, the coherent state appears as in Fig. 17.2a, while a squeezed state appears as in Fig. 17.2b.

Given a field described by the annihilation operator $a$, we form two hermitian conjugate operators giving its two quadratures as

$$d_1 = \frac{1}{2} (ae^{i\phi} + a^\dagger e^{-i\phi}) , \quad d_2 = \frac{1}{2i} (ae^{i\phi} - a^\dagger e^{-i\phi}) ,$$  \hspace{1cm} (17.3)

with $[d_1, d_2] = i/2$, so that $\Delta d_1 \Delta d_2 \geq 1/4$. These two operators correspond to position and momentum for the case of a mechanical oscillator. The variance $\Delta d_1^2$ is given by

$$\Delta d_1^2 = \langle d_1^2 \rangle - \langle d_1 \rangle^2 .$$  \hspace{1cm} (17.4)

Consider for a moment a quantum state such that the expectation value of the electric field is zero, $\langle a \rangle = \langle a^\dagger \rangle = \langle d_i \rangle = 0$. This reduces $\Delta d_1^2$ to

$$\Delta d_1^2 = \frac{1}{4} \left[ \langle a^\dagger a \rangle + \langle aa^\dagger \rangle + \langle (a^2) e^{2i\phi} + c.c. \rangle \right]$$

$$= \frac{1}{4} + \frac{1}{2} \langle a^\dagger a \rangle + \frac{1}{2} \text{Re} \{ \langle aa \rangle e^{2i\phi} \} .$$  \hspace{1cm} (17.5)