Chapter 3
Inner-Product Vector Spaces (Euclidean and Unitary Spaces)

Applications in physics of vector spaces $V_n(\mathbb{R})$ or $V_n(\mathbb{C})$ is insignificant, since in them we cannot define measurable quantities like lengths or angles.

3.1 Euclidean Spaces $E_n$

In $\mathbb{R}^3$, we were able to do this by making use of the dot product: if $\vec{x} = [x_1, x_2, x_3]^T$ and $\vec{y} = [y_1, y_2, y_3]^T$ are two vectors (matrix-columns of three real numbers) in $\mathbb{R}^3$, then their dot product is defined as

$$\vec{x} \cdot \vec{y} = x_1y_1 + x_2y_2 + x_3y_3 = \sum_{i=1}^{3} x_iy_i.$$ 

As a matter of fact, the dot product of two geometric vectors (arrows) $\vec{x}$ and $\vec{y}$ was originally introduced, as it is done in physics, as the product of their lengths with the cosine of the smaller angle $\alpha$ between them:

$$\vec{x} \cdot \vec{y} = |\vec{x}| |\vec{y}| \cos \alpha \quad \text{(the original definition)}.$$

It is the product of the length $|\vec{x}|$ of $\vec{x}$ with the (positive or negative) length of projection $|\vec{y}| \cos \alpha$ of $\vec{y}$ along the line of $\vec{x}$ or the other way round $|\vec{y}|$ times the
length of projection $|\vec{x}| \cos \alpha$. Remember that in physics the work done by a force $\vec{F}$ producing a displacement $\vec{d}$ is
\[
\vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}| \cos \alpha.
\]

The four most important properties of the dot product

1. $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$—symmetry or commutative property (obvious);
2. $(\vec{x}_1 + \vec{x}_2) \cdot \vec{y} = \vec{x}_1 \cdot \vec{y} + \vec{x}_2 \cdot \vec{y}$—distributive property with regard to the vector addition in the first factor (the projection of $\vec{x}_1 + \vec{x}_2$ along the line of $\vec{y}$ is the sum of the projections of $\vec{x}_1$ and $\vec{x}_2$, see the figure);
3. $a(\vec{x} \cdot \vec{y}) = (a \cdot \vec{x}) \cdot \vec{y} = \vec{x} \cdot (a \cdot \vec{y})$, $a \in \mathbb{R}$—associative property with respect to multiplication with scalars

Due to the symmetry, it is also valid for the second factor $\vec{x} \cdot (\vec{y}_1 + \vec{y}_2) = \vec{x} \cdot \vec{y}_1 + \vec{x} \cdot \vec{y}_2$.

4. $(\vec{x} \cdot \vec{x}) = |\vec{x}|^2 > 0$ if $\vec{x} \neq \vec{0}$; $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 0$ iff $\vec{x} = \vec{0}$—positive definite property [only the zero vector $\vec{0}$ has zero length, others have positive lengths, $|\vec{x}|^2 = 0 \Leftrightarrow |\vec{x}| = 0 \Leftrightarrow \vec{x} = \vec{0}$];