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κ-Compatiblε Tessellations*

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Summary. The vast majority of visualization algorithms for finite element (FE) simulations assume that linear constitutive relationships are used to interpolate values over an element, because the polynomial order of the FE basis functions used in practice has traditionally been low – linear or quadratic. However, higher order FE solvers, which become increasingly popular, pose a significant challenge to visualization systems as the assumptions of the visualization algorithms are violated by higher order solutions. This paper presents a method for adapting linear visualization algorithms to higher order data through a careful examination of a linear algorithm’s properties and the assumptions it makes. This method subdivides higher order finite elements into regions where these assumptions hold (κ-compatibility). Because it is arguably one of the most useful visualization tools, isosurfacing is used as an example to illustrate our methodology.

1 Introduction

People have been approximating solutions to partial differential equations (PDEs) ever since PDEs were conceived. Much more recently, a class of techniques known as hp-adaptive methods have been developed in an effort to converge to a solution faster than previously possible, or to provide more accurate approximations than traditional finite element simulation within the same amount of computational time. These new techniques can increase both the hierarchical (h) and polynomial (p) levels of detail – or degrees of freedom – during a simulation.

Once these solution approximations have been computed, they must be characterized in some way so that humans can understand and use the results. This paper develops a technique for partitioning higher-order cells in order to

*This work was supported by the United States Department of Energy, Office of Defense Programs, Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed-Martin Company, for the United States Department of Energy under contract DE-AC04-94-AL85000.
characterize the behavior of their geometric and scalar field curvatures during post-processing. We say that such partitions are \( \kappa \)-compatible. While a past paper [8] has presented our software framework for creating these partitions, this paper presents a full description of the algorithm and a rigorous proof of the conditions under which it will work and terminate.

Currently, visualization techniques for quadratic and higher order FE solutions are very limited in scope and/or cannot guarantee that all topological features are captured [1, 3, 7, 8, 6]. These techniques are also limited in their applicability to a subset of visualization techniques. Moreover, although some production-level tools currently offer support for quadratic elements, they do not always do so correctly (cf. [8] for a discussion); in the case of isocontouring, consider for example the following scalar field:

\[
\Phi_1 : [-1, 1]^3 \rightarrow \mathbb{R} \\
(x, y, z)^T \mapsto x^2 + y^2 + 2z^2,
\]

interpolated over a single Q2 Lagrange element (hexahedron with degree 2 Lagrange tensor-product interpolation over 27 nodes), with linear geometry, where one is interested in the isocontours \( \Phi_1^{-1}(1) \) and \( \Phi_1^{-1}(2) \).

As shown in Figure 1, left, the linear isocontouring approach (implemented here in ParaView) completely misses \( \Phi_1^{-1}(1) \), because it is entirely contained within the cell. Meanwhile, the topology of \( \Phi_1^{-1}(2) \) is correct, but its geometry is poorly captured. On the other hand, our new method (Figure 1, right) captures the correct topologies of both isocontours, and provides a much better geometric approximation of \( \Phi_1^{-1}(2) \) than linear isocontouring does.

The lack of tools that are applicable to most visualizations of higher order element simulations, and that can guarantee correctness of the results, prevents analysts from exploiting such simulations. In this paper, we propose a solution to this problem.

**Fig. 1.** Isocontours of \( \Phi_1 \) for the isovalues 1 (cyan) and 2 (green): linear isocontouring approach (left), and our new topology-based approach (right).