Efficient Delaunay Mesh Generation from Sampled Scalar Functions

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Abstract: Many modern research areas face the challenge of meshing level sets of sampled scalar functions. While many algorithms focus on ensuring geometric qualities of the output mesh, recent attention has been paid to building topologically accurate Delaunay conforming meshes of any level set from such volumetric data.

In this paper, we present an algorithm which constructs a surface mesh homeomorphic to the true level set of the sampled scalar function. The presented algorithm also produces a tetrahedral volumetric mesh of good quality, both interior and exterior to the level set. The meshing scheme substantially improves over the existing algorithms in terms of efficiency. Finally, we show that when the unknown sampled scalar function, for which the level set is to be meshed, is approximated by a specific class of interpolant, the algorithm can be simplified by taking into account the nature of the interpolation scheme so as to circumvent some of the critical computations which tend to produce numerical instability.

1 Problem and Motivation

A wide variety of science and engineering applications rely on accurate level set triangulation. This is especially true for multiscale models in biology, such as macromolecular structures extracted from reconstructed single particle cryo-EM (Electron Microscopy), cell-processes and cell-organelles extracted from TEM (Tomographic Electron Microscopy), and even trabecular bone models extracted from SR-CT (Synchrotron Radiation Micro-Computed Tomography) imaging. Computational analysis of these models for estimation of nano, micro, or mesoscopic structural properties depends on the mesh representation of the contour components respecting their topological features.
Our goal is to find an algorithm to solve the following problem. The input to the algorithm is a rectilinear sampling of a bounded domain of an unknown scalar function $F$. The rectilinear grid need not be uniform; it may be adaptive as in the case of an octtree. The user then specifies a local interpolant to generate a level set approximation for any isovalue $v$; we use $\Sigma$ to denote this level set of the interpolating function. Since the function $F$ is unknown, we must assume that the local interpolant produces a good approximation of the function $F$ within each cell of the grid. Our algorithm is general enough to use any local interpolant, however, in our experience, a trilinear interpolant is the most natural choice.

Our goal is construct a mesh in an efficient manner such that the following properties hold:

1. **Topological Guarantee:** $M$ is homeomorphic to $\Sigma$. 

Fig. 1. Various stages of our algorithm. (a) A rectilinear grid with sample values of an unknown function at the grid points. Within cells, the function is approximated with a trilinear interpolant. For the purpose of visualization only, we collect a set of points (green) on the surface and display them. A narrow region of the surface is magnified below. (b) Another view of the data (right) and the same view of the mesh generated by Marching Cubes [22]. Note that the mesh is disconnected in the thin region. (c) The mesh generated by the restricted Delaunay triangulation of only edge and grid points. Blue facets have a grid point as at least one of their vertices. This point set is still not sufficient to produce a Delaunay-conforming mesh. (d) The mesh generated by addition of sample points of $\Sigma$. The topology is now recovered (Property I). (e,f) Geometrical refinement for progressively smaller value of $\epsilon$. (g) Even in the magnified portion of the thin region, the triangles approximate the geometry nicely (Property II). (h) All the points involved in construction of the mesh including grid points (blue), edge points (green), and new points added by the algorithm (red). Observe that in order to recover the topology and reduce the geometric error in the approximation, many surface sample points are added to the point set.