Three Important Theorems for Flow Stability

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Abstract: A criterion of flow instability and turbulent transition in curved shear flows is obtained via the analysis of energy equilibrium of fluid particles. Then, three important theorems for fluid stability are deduced: (1) Potential flow (inviscid and irrotational) is stable. (2) Inviscid rotational (inviscid and nonzero vorticity) flow is unstable. (3) Velocity profile with an inflectional point is unstable when there is no work input or output to the system, for both inviscid and viscous flows. These theorems have significant implications for vortex breakdown and flow transition.

Key words: theorem, flow instability, turbulent transition, shear flows, energy gradient, energy loss

INTRODUCTION

The mechanisms of flow instability and turbulent transition are still not fully understood although much progress has been achieved in the past century [1]. The classical linear stability theory, the energy method based on Reynolds-Orr equation, the weakly nonlinear method, and the secondary instability theory have been developed to predict and to study the above mentioned phenomena. These theories help us to understand the problem of flow instability. However, there is still discrepancy between the theories and experiments [1].

In a recent study, Dou developed an “Energy Gradient Theory” by rigorous derivation, in which the detail of amplification or decay of the disturbance in parallel flows has been described [2,3]. The theory proposes that in shear flows it is the transverse energy gradient interacting with a disturbance to lead to the flow instability, while the energy loss, due to viscous friction along the streamline, damps the disturbance. The mechanisms of velocity inflection and formation and lift of the hairpin vortex are well explained with the analytical result. It is shown that the disturbed particle exchanges energy with other particles in transverse direction during the cycle and causes the particle leaves its equilibrium position. The analysis demonstrated that the threshold amplitude of disturbance for transition to turbulence is scaled with Re by an exponent of γ = −1 in parallel flows, which explains the recent experimental result of pipe flow by Hof et al. [4]. The study also confirms the results from asymptotic analysis (for Re → ∞) of the Navier-Stokes equations by Chapman [5]. Finally, Dou obtained a criterion of stability which is expressed as a function of K and the disturbance amplitude. Here, K is a function of energy gradient and energy loss in the flow field. For a given base flow, the maximum of this function in the flow field, K_{max}, is taken as a stability parameter [2,3]. This approach obtains a consistent value of K_{max} for the critical condition (i.e., at minimum Reynolds number) of turbulent transition in parallel flows including plane Poiseuille flow, pipe Poiseuille flow and plane Couette flow [2,3]. In this paper, the energy gradient theory is extended to curved flows. Then, based on the results, three important theorems for flow stability are deduced.

ENERGY GRADIENT THEORY APPLIED TO CURVED FLOWS

The energy gradient theory has been described for parallel flows in detail in [2,3]. The argument is started with the consideration of elastic collision of particles when a disturbance is imposed to the base of a parallel shear flow (Figure 1). A fluid particle P at its equilibrium position will move a cycle in vertical direction under a vertical disturbance, and it will have several collisions with the particles in a period. For parallel flows, only kinetic energy difference exists between neighboring streamlines. When fluid particles exchange energy by collisions, it is the exchange of the kinetic energy. For a cycle of disturbances, the fluid particle may absorb energy by collision in the first half-period and it may...
release energy in the second half-period because of the gradient of the total mechanical energy. The total momentum and total mechanical energy are conserved during the elastic collisions. In addition, there exists energy loss due to viscous friction between fluid layers. The stability of the particle can be related to the energy gained by the particle through vertical disturbance and the energy loss due to viscosity along streamline in a half-period. After the particle moves a half cycle, if the net energy gained by collisions is zero, this particle will stay in its original equilibrium position (streamline). If the net energy gained by collisions is larger than zero, this particle will be able to move into equilibrium with a higher energy state. If the collision in a half-period results in a drop of total energy, the particle can move into lower energy equilibrium. However, there is a critical value of energy increment which is balanced (damped) by the energy loss due to viscosity. When the energy increment accumulated by the particle is less than this critical value, the particle could not leave its original equilibrium position after a half-cycle. Only when the energy increment accumulated by the particle exceeds this critical value, could the particle migrate to its neighbor streamline and its equilibrium will become unstable. For parallel flows, it is shown that the relative magnitude of the energy gained from collision ($\Delta E$) and the energy loss due to viscous friction ($\Delta H$) determines the disturbance amplification or decay. It has been shown that energy loss damps flow disturbance and enhance stability [7]. Thus, for a given flow, a stability criterion can be written as below for the half-period,

$$F = \frac{\Delta E}{\Delta H} = \left( \frac{\partial E}{\partial n} \right) / \left( \frac{\partial H}{\partial s} \right) u \equiv \frac{2}{\pi} K \frac{\Delta \omega}{u} = \frac{2}{\pi} K \frac{\psi_m}{u} < \text{const}$$  \hspace{1cm} (1)

and

$$K = \frac{\partial E / \partial n}{\partial H / \partial s}$$  \hspace{1cm} (2)

Here, $F$ is a function of coordinates which expresses the ratio of the energy gained in a half-period by the particle and the energy loss due to viscosity in the half-period; $K$ is a dimensionless field variable (function) and expresses the ratio of transversal energy gradient and the rate of the energy loss along the streamline; $\psi_m = \Delta \omega$ is the amplitude of disturbance velocity and the disturbance has a period of $T = 2\pi / \omega$; $E = (1/2)pV^2$ is the kinetic energy per unit volume of fluid, $s$ is along the streamwise direction and $n$ is along the transverse direction.

For curved flows, the difference of energy between streamlines is the difference of the total mechanical energy. Extending the theory from parallel flow to curved flow (Figure 1), we only need to change the kinetic energy ($E = (1/2)m u^2$) to the total mechanical energy ($E = p + (1/2)pV^2$), and to make the velocity ($u$) along the streamline. We use the ($s$, $n$) to express the coordinates in streamwise and transverse directions, respectively. Using the similar derivations to those in [3], the energy variation of per unit volume of fluid for a half-period for the disturbed fluid particles can be obtained. The equation is the same as Eqs.(1) and (2) except $E = p + (1/2)pV^2$.

It is found from Eq.(1) that the instability of a flow depends on the value of $K$ and the amplitude of the relative disturbance velocity $\psi_m / u$. For given disturbance, the maximum of $K$, $K_{\text{max}}$, in the flow domain determines the stability. Therefore, $K_{\text{max}}$ is taken as a stability parameter here. For