Convection in a Fluid Layer Heated from below and Subjected to Time Periodic Horizontal Accelerations

J. Tao1, W. Pesch2, F. H. Busse2

1LTCS and Department of Mechanics and Aerospace Engineering, Peking University, Beijing, 1000871 China
2Institute of Physics, University of Bayreuth, D-95440 Bayreuth, Germany
Email: jtao@pku.edu.cn

Abstract A theoretical study is presented of convection in a horizontal fluid layer heated from below which is periodically accelerated in its plane. The analysis is based on Galerkin methods of the underlying Boussinesq equations. Shaking in a fixed direction breaks the original isotropy of the layer. At onset of convection we find longitudinal rolls, where the roll axis aligns parallel to the acceleration direction. With increasing acceleration amplitude a shear instability takes over and transverse rolls with the axis perpendicular to the shaking direction nucleate at onset.

Key words: convection, instability

INTRODUCTION

The topic of convection in fluid layers heated from below or from above that is subjected to time periodic acceleration has received considerable attention in recent years. The book by Gershuni and Lyubimov[1] provides a good overview of the numerous cases that are obtained as the direction of acceleration and the inclination of the layer with respect to the horizontal are varied. A special case is the horizontal fluid layer subjected to horizontal accelerations which is the issue of this paper. This problem has first been considered by Gershuni et al. [2] for a fixed direction of the acceleration and for the single value P=1 of the Prandtl numberP in the linear regime.

In the present paper we extend these investigations by studying the influence of variations of P on the onset of convection. In section 2 we shall first discuss the mathematical formulation of our problem and sketch briefly the numerical methods and in section 3 a linear stability analysis of the homogeneous basic state is used to describe the properties of the system at onset of convection. The paper closes with a concluding discussion in section 4.

MATHEMATICAL BACKGROUND

We consider a horizontal fluid layer heated from below that is subjected to a harmonic acceleration with frequency $\Omega$ of the form

$$g(t) = -g(e_z + \frac{b_x}{g}e_x \sin \Omega t + \frac{b_y}{g}e_y \cos \Omega t)$$

(2.1)

where $-ge_z$ is the acceleration of gravity opposite to the direction of the vertical unit vector $e_z$. The additional terms are caused by a periodic shaking with amplitudes $b_x$ and $b_y$, respectively, of the fluid layer in the horizontal directions described by the unit vectors $e_x$ and $e_y$. We employ the Boussinesq approximation in that all material properties are regarded as constant, except the temperature dependence of the density described by $\rho = \rho_0(1 - \alpha(T - T_0))$ in the acceleration terms. $T_0$ is defined as the average of the temperatures $T_1$ and $T_2$ prescribed at the upper and lower boundaries of the layer, respectively. We focus the attention on systems with a large aspect ratio, where the lateral dimensions are much larger than the layer height $d$. 
We use $d$ as length scale and the vertical diffusion time $t_v = d^2/\kappa$ as time scale where $\kappa$ is the thermal diffusivity of the fluid. Furthermore $\nu \kappa/\alpha gd^3$ is chosen as scale for the deviation $\Theta$ of the temperature from its purely conductive state where $\nu$ is the kinematic viscosity. We arrive thus at the following basic equations in dimensionless form

\[ p^{-1} D_t \nu = -\nabla \cdot \left( e_x G_x z \cot \omega t + e_y G_y \cos \omega t \right) (\theta - R_z) + e_x \Theta + \nabla^2 \nu \]  
(2.2a)

\[ D_t \Theta = Re_z \cdot \nu + \nabla^2 \Theta, \quad \text{with} \quad D_t = \frac{\partial}{\partial t} + (\nu \cdot \nabla) \]  
(2.2b)

\[ \nabla \cdot \nu = 0 \]  
(2.2c)

Here the Rayleigh number $R$, the Prandtl number $P$, the dimensionless angular frequency $\omega$ and acceleration parameters $G_x$ and $G_y$ are defined by

\[ R = \frac{\alpha g}{\nu K} (T_2 - T_1) d^3, \quad P = \frac{\nu}{K}, \quad \omega = \frac{\Omega d^2}{K}, \quad G_x = \frac{b_x \Omega^2}{\nu}, \quad G_y = \frac{b_y \Omega^2}{\nu} \]  
(2.3)

All gradient terms on the right hand side of (2.2a) have been collected into the pressure $\Pi$.

**LINEAR ANALYSIS**

For simplicity, a symbolic notation is used in the following

\[ C_{\frac{\partial}{\partial t}} V(x, z, t) = L V(x, z, t), \quad \text{with} \quad L = A + RB \]

where the symbolic vector $V(x, z, t)$ represents the different fields in our problem and $x = (x, y)$. The ground state corresponds to $V(x, z, t) = 0$. The operator $C$, $L$ are linear differential operators and $A$, $B$ contain contributions periodic in time with frequency $\omega$.

As a consequence of the explicit periodic time dependence of the linear operator the linearized equations are solved with the Floquet ansatz:

\[ V(x, z, t) = \exp[\sigma t + ig \cdot x] V_{\text{lin}}(q, z, t); \quad q = (q, p), \quad x = (x, y) \]  
(3.1)

with $V_{\text{lin}}(q, z, t) = V_{\text{lin}}(q, z, t + T)$. Thus we arrive at the linear eigenvalue problem:

\[ \sigma(q, R) CV_{\text{lin}}(q, z, t) = (A + RB - C_{\frac{\partial}{\partial t}}) V_{\text{lin}}(q, z, t) \]  
(3.2)

We are interested in the growth rate, $\sigma_0(q, R)$, i.e. the eigenvalue $\sigma(q, R)$ with the largest real part. The condition $\text{Re}[\sigma_0(q, R)] = 0$ yields the neutral curve $R_0(q)$ with its minimum values $R_c$ (the critical Rayleigh number) at the critical wave vector $q_c$.

Eq. (3.2) is solved by the Galerkin method, which includes, for instance, the following ansatz for the temperature component, $\Theta_{\text{lin}}$, of $V_{\text{lin}}(q, z, t)$:

\[ \Theta_{\text{lin}}(q, z, t) = \sum_{n=1}^{N} \sum_{k=-K}^{K} \theta_{\text{lin}}(q, n|k) \exp[i k \omega t] \sin n \pi (z + 1/2) \]  
(3.3)

The analogous expansion for the $f$ component of $V_{\text{lin}}$ is characterized by the expansion coefficients $f_{\text{lin}}(q, n|k)$. Thus we arrive at a linear algebraic eigenvalue problem, which yields $\sigma_0(q, R)$ and the corresponding expansion coefficients $\theta_{\text{lin}}$, $f_{\text{lin}}$ of the linear eigenvector. The summations in (3.3) have to be truncated; typically a truncation at $K$, $N = 10$ is well sufficient for numerical accuracy. In the following we will investigate the case of unidirectional shaking ($G_y = 0$).

There are two distinguished types of solutions of the form (3.1):

- **Longitudinal roll solutions with** $q = (0, p)$ **which are time independent and where the neutral curve** $R_0(q)$ **and thus its minimum** $R_0(q_c) = R_0 = 1707.76$ **as well as the critical wave vector** $q_c = (0, p_c)$ **with** $p_c = 3.116$ **do not depend on the system parameters. This solution is thus identical with**