Chapter 14
Describing Function Analysis

Under some circumstances, a control valve suffering from stiction induces a limit-
cycle oscillation in the control loop. This chapter gives an analysis to gain insights
into the conditions under which such limit cycles might arise and also provides
some understanding of how the magnitude and frequency of the limit cycle are in-
fluenced by tuning of the controller. Describing function analysis uses a quasilinear
approach in which the nonlinearity is approximated by a signal-dependent linear
gain. The key assumption, which is met in the systems considered here, is that the
behaviour of the control loop can be understood by considering just the fundamental
sinusoidal component of the limit cycle. Limit-cycle oscillations caused by valve
stiction can be studied using describing function analysis, and valuable insights
about the signal-generating system can be obtained. The analysis of this chapter
focuses on the describing function of the two-parameter stiction model developed
in Chap. 13.

14.1 Introduction

A nonlinear actuator with a stiction characteristic may cause limit cycling in a con-
trol loop. Further insights into the behaviour of such systems may be achieved
through a describing function analysis (Cook, 1986). The nonlinearity is modelled
by a nonlinear gain $N$. The assumptions inherent in the approximation are that (1)
there are periodic signals present in the system and (2) the controlled system is low
pass and responds principally to the fundamental Fourier component. The condi-
tions for oscillation in a negative feedback loop arise when the loop gain is $-1$:

$$G_o(i\omega) = -\frac{1}{N(X_m)} \quad (14.1)$$

where $G_o(i\omega)$ is the open-loop frequency response that includes the controlled sys-
tem and the proportional plus integral controller, and $N(X_m)$ is the describing func-
tion that depends on the magnitude of the controller output $X_m$. When the condition
$G_o(i\omega) = -1/N(X_m)$ is met, the system will oscillate spontaneously with a limit
cycle. The variation of the quantity $-1/N(X_m)$ with signal amplitude means that signals initially present in the loop as noise can grow until they are large enough to satisfy the equality and hence provide a self-starting oscillation. The solution to the complex equation $G_o(i\omega) = -1/N(X_m)$, if one exists, may be found graphically by superposing plots of $G_o(i\omega)$ and $-1/N$ on the same set of axes.

14.2 Describing Function Analysis for Two-Parameter Stiction Model

The describing function of a nonlinearity is:

$$N = \frac{Y_f}{X}$$  \hspace{1cm} (14.2)

where $X$ is a harmonic input to the nonlinearity of angular frequency $\omega_o$ and $Y_f$ is the fundamental Fourier component at angular frequency $\omega_o$ of the output from the nonlinearity. Thus, a Fourier analysis is needed on the output signals shown as bold lines in Fig. 14.1a. The quantity $N$ depends upon the magnitude of the input $X_m$. $N$ is complex for the stiction nonlinearity because the output waveform has a phase lag compared to the input.

14.2.1 Derivation of the Describing Function

The input–output behaviour of stiction nonlinearity is shown in Fig. 14.1. The output from stiction nonlinearity (i.e. the solid line in Fig. 14.1a) is not analytic. The term $d$ in this figure is used in place of $(S-J)$, which is equal to the deadband. It is useful to consider a sine wave input (dotted line in Fig. 14.1a) with angular frequency of 1 rad·s$^{-1}$ and period $2\pi$. The output (the solid line in Fig. 14.1a) is then:

$$y(t) = \begin{cases} 
  k \left( X_m \sin(t) - \frac{S-J}{2} \right) & 0 \leq t \leq \frac{\pi}{2} \\
  k \left( X_m - \frac{S-J}{2} \right) & \frac{\pi}{2} \leq t \leq \pi - \phi \\
  k \left( X_m \sin(t) + \frac{S-J}{2} \right) & \pi - \phi \leq t \leq \frac{3\pi}{2} \\
  k \left( -X_m + \frac{S-J}{2} \right) & \frac{3\pi}{2} \leq t \leq 2\pi - \phi \\
  k \left( X_m \sin(t) - \frac{S-J}{2} \right) & 2\pi - \phi \leq t \leq 2\pi 
\end{cases}$$

where $X_m$ is the amplitude of the input sine wave, $S$ the deadband plus stickband, $J$ the slip–jump, $\phi = \sin^{-1}\left( \frac{X_m - S}{X_m} \right)$ and $k$ the slope of the input–output characteristic in the moving phase. $k$ is assumed to be 1 for a valve with linear characteristics.