Chapter 7
Non-linear Post Buckling Finite Element Analysis of Plates and Shells

7.1 Introduction

In this chapter we present the nonlinear post-buckling analysis of plates and shells by the finite element method, using the four-node $C^0$ strain element. As previously, the element tangent stiffness matrix is given explicitly, i.e., without any numerical integration, which results in a computationally efficient algorithm.

The buckling behavior of plates and shells differs from that of columns. A slender straight column subjected to gradually increasing compressive loads at its ends, becomes unstable and buckles when the applied loads reach a certain value called the critical load. The column, in general, collapses completely when a slight incremental load is applied beyond the critical load. The load-carrying capacity of plates and shells, however, differs significantly from the critical loads predicted by the stability analysis. This can be demonstrated by large discrepancies between results given by the classical stability theory of plates and shells and by experimental observations. Von Karman and Tsien (1939) first showed that such buckling behavior of plates and shells is caused by the highly unstable and nonlinear post-buckling phenomenon exhibited by thin structures. The nonlinear post-buckling analysis of plates and shells is necessary for correct prediction of the load-carrying capacity of these structures. As it is difficult to obtain rigorous analytical solutions of nonlinear post-buckling problems in most cases, numerical methods must be adopted.

Because of the increased use of shells and plates in the aerospace and nuclear industries, instability analysis of these structures has become more important in recent years. With the advent of computers and the finite element technique, the engineer’s ability to solve complex structural problems has greatly improved. Numerous finite element models for the large displacement and post-buckling analysis of plates and shells have been suggested (Murray and Willson, 1969; Gallagher and Thomas, 1973; Sabir and Lock, 1973; Wood and Zienkiewicz, 1977; Horrigmoe and Bergan, 1978; Bathe and Dvorkin, 1986; Noor et al., 1989; Kim and Voyiadjis, 1999). In most nonlinear finite elements, however, the element matrices are evaluated through numerical integration, which is computationally expensive, especially in nonlinear analysis where the element matrices must be evaluated numerous times.
An assumed strain shell element, with five degrees of freedom at each node, was developed for nonlinear post buckling analysis by Voyiadjis and Shi (1992). The element formulation is based on the Updated Lagrangian description, the von Karman assumption, and the quasi-conforming element method (an assumed strain method – Tang et al., 1980, 1983). The four-node quadrilateral element can reduce to the corresponding three-node triangular element.

The formulation of the explicit element tangent stiffness matrix is similar to that presented in Chap. 4. The emphasis here is on evaluation of the initial surface coordinates for large deformation analysis of shells. The post-buckling analysis using our algorithm is performed on a number of structures and compared with the existing analytical/numerical solutions where they are available. These numerical examples demonstrate that our nonlinear plate element is efficient and accurate.

### 7.2 Element Tangent Stiffness Matrix

The element considered here is the four-node quadrilateral strain element; each node has five degrees of freedom, three translations, and two rotations. The element formulation for the nonlinear analysis of plates and shells was presented in the previous chapters. Here we use this element for the post-buckling analysis of plates and shells. For convenience, we repeat some of the equations derived in previous chapters.

#### 7.2.1 Element Stiffness in Local Coordinates

The von Karman assumptions and Updated Lagrangian description are used here to derive the element tangent stiffness matrix. In the finite element modeling of transverse shear deformable plates using a generalized displacement method, the incremental bending strains $\varepsilon_b$, membrane strains $\varepsilon_m$, transverse shear strains $\varepsilon_s$, and slopes $\theta$ of an element defined in the element local coordinates take the form (Shi and Voyiadjis, 1990, 1991a,c):

$$
\varepsilon_b = \begin{bmatrix}
\frac{\partial \Delta \phi_x}{\partial x} \\
\frac{\partial \Delta \phi_y}{\partial x} \\
\frac{\partial \Delta \phi_x}{\partial y} + \frac{\partial \Delta \phi_y}{\partial x}
\end{bmatrix} = B_b \Delta q_e
$$

(7.1)