Chapter 13
Bökstedt–Neeman Resolutions
and HyperExt Sheaves

(13.1) Let $T$ be a triangulated category with small direct products. Note that a direct product of distinguished triangles is again a distinguished triangle (Lemma 3.1).

Let

$$\cdots \to t_3 \xrightarrow{s_3} t_2 \xrightarrow{s_2} t_1$$

be a sequence of morphisms in $T$. We define $d : \prod_{i \geq 1} t_i \to \prod_{i \geq 1} t_i$ by $p_i \circ d = p_i - s_{i+1} \circ p_{i+1}$, where $p_i : \prod_{i} t_i \to t_i$ is the projection. Consider a distinguished triangle of the form

$$M \xrightarrow{m} \prod_{i \geq 1} t_i \xrightarrow{d} \prod_{i \geq 1} t_i \xrightarrow{q} \Sigma M,$$

where $\Sigma$ denotes the suspension.

We call $M$, which is determined uniquely up to isomorphisms, the homotopy limit of (13.2) and denote it by $\text{holim} t_i$.

(13.3) Dually, homotopy colimit is defined and denoted by $\text{hocolim}$, if $T$ has small coproducts.

(13.4) Let $\mathcal{A}$ be an abelian category which satisfies (AB3*). Let $(F_\lambda)_{\lambda \in \Lambda}$ be a small family of objects of $K(\mathcal{A})$. Then for any $G \in K(\mathcal{A})$, we have that

$$\text{Hom}_{K(\mathcal{A})}(G, \prod_{\lambda} F_\lambda) = H^0(\text{Hom}_\bullet(G, \prod_{\lambda} F_\lambda)) \cong H^0(\prod_{\lambda} \text{Hom}_\bullet(G, F_\lambda))$$

$$\cong \prod_{\lambda} H^0(\text{Hom}_\bullet(G, F_\lambda)) = \prod_{\lambda} \text{Hom}_{K(\mathcal{A})}(G, F_\lambda).$$

That is, the direct product $\prod_{\lambda} F_\lambda$ in $C(\mathcal{A})$ is also a direct product in $K(\mathcal{A})$.

(13.5) Let $\mathcal{A}$ be a Grothendieck abelian category, and $(t_\lambda)$ a small family of objects of $D(\mathcal{A})$. Let $(F_\lambda)$ be a family of $K$-injective objects of $K(\mathcal{A})$ such that $F_\lambda$ represents $t_\lambda$ for each $\lambda$. Then $Q(\prod_{\lambda} F_\lambda)$ is a direct product of $t_\lambda$ in

Lemma 13.6. Let $I$ be a small category, $S$ be a scheme, and let $X_\bullet \in \mathcal{P}(I, \mathcal{S}_\mathcal{C}/S)$. Let $\mathbb{F}$ be an object of $\mathcal{C}(\text{Mod}(X_\bullet))$. Assume that $\mathbb{F}$ has locally quasi-coherent cohomology groups. Then the following hold.

\begin{enumerate}[i]
    \item Let $\mathcal{I}$ denote the full subcategory of $\mathcal{C}(\text{Mod}(X_\bullet))$ consisting of bounded below complexes of injective objects of $\text{Mod}(X_\bullet)$ with locally quasi-coherent cohomology groups. There is an $\mathcal{I}$-special inverse system $(I_n)_{n \in \mathbb{N}}$ with the index set $\mathbb{N}$ and an inverse system of chain maps $(f_n : \tau_{\geq -n} \mathbb{F} \to I_n)$ such that
        \begin{enumerate}
            \item $f_n$ is a quasi-isomorphism for any $n \in \mathbb{N}$.
            \item $I_n^i = 0$ for $i < -n$.
        \end{enumerate}
    \item If $(I_n)$ and $(f_n)$ are as in 1, then the following hold.
        \begin{enumerate}
            \item For each $i \in \mathbb{Z}$, the canonical map $H^i(\lim I_n) \to H^i(I_n)$ is an isomorphism for $n \geq \max(1, -i)$, where the projective limit is taken in the category $\mathcal{C}(\text{Mod}(X_\bullet))$, and $H^i(\cdot)$ denotes the $i$th cohomology sheaf of a complex of sheaves.
            \item $\lim f_n : \mathbb{F} \to \lim I_n$ is a quasi-isomorphism.
            \item The projective limit $\lim I_n$, viewed as an object of $K(\text{Mod}(X))$, is the homotopy limit of $(I_n)$.
            \item $\lim I_n$ is $K$-injective.
        \end{enumerate}
\end{enumerate}

Proof. The assertion 1 is [39, (3.7)].

We prove 2, i. Let $j \in \text{ob}(I)$ and $U$ an affine open subset of $X_j$. Then for any $n \geq 1$, $I_n^i$ and $H^i(I_n)$ are $\Gamma((j, U), \cdot)$-acyclic for each $i \in \mathbb{Z}$. As $I_n$ is bounded below, each $Z^i(I_n)$ and $B^i(I_n)$ are also $\Gamma((j, U), \cdot)$-acyclic, and the sequence

$$0 \to \Gamma((j, U), Z^i(I_n)) \to \Gamma((j, U), I_n^i) \to \Gamma((j, U), B^{i+1}(I_n)) \to 0 \quad (13.7)$$

and

$$0 \to \Gamma((j, U), B^i(I_n)) \to \Gamma((j, U), Z^i(I_n)) \to \Gamma((j, U), H^i(I_n)) \to 0 \quad (13.8)$$

are exact for each $i$, as can be seen easily, where $B^i$ and $Z^i$ respectively denote the $i$th coboundary and the cocycle sheaves.

In particular, the inverse system $(\Gamma((j, U), B^i(I_n)))$ is a Mittag-Leffler inverse system of abelian groups by (13.7), since $(\Gamma((j, U), I_n^i))$ is. On the other hand, as we have $H^i(I_n) \cong H^i(\mathbb{F})$ for $n \geq \max(1, -i)$, the inverse system $(\Gamma((j, U), H^i(I_n)))$ stabilizes, and hence we have $(\Gamma((j, U), Z^i(I_n)))$ is also Mittag-Leffler.

Passing through the projective limit,

$$0 \to \Gamma((j, U), Z^i(\lim I_n)) \to \Gamma((j, U), \lim I_n) \to \Gamma((j, U), \lim B^{i+1}(I_n)) \to 0$$