In this Chapter, the problem of trajectory planning in 3D is addressed. This problem is of relevance in operations with multi-degrees of freedom machines, like for example robots or special sanding/milling machines. Planning motions in the 3D space is more complex than in the single axis case, since in general two aspects must be defined: the geometry of the trajectory (e.g. a straight line, a circle, and so on), and the motion law to be adopted while following the geometric path. Moreover, also the problem of the orientation (of the tool) has to be considered. Therefore, there are at least seven variables to be specified for a 3D trajectory: three for the position, three for the orientation and one for the motion law.

Besides these aspects, in order to execute a desired motion in the space by means of a multi-degree-of-freedom machine, it is also necessary to consider the fact that a (inverse) kinematic model of the machine must be used in order to transform the given trajectory from the 3D space, where it is specified, to the space of actuation (usually called the joint space) where the motion of the actuators takes place. In general, this kinematic transformation may be quite complex and depends on the particular machine or robot at hand.

8.1 Introduction

The definition of a trajectory in the Cartesian (3D) space implies the determination of a geometric path to be tracked with a prescribed motion law, that in turns can be defined by means of functions similar to those reported in the previous chapters. For this purpose, it is convenient to consider a parametric representation of a curve in the space

\[ p = p(u), \quad u \in [u_{\text{min}}, u_{\text{max}}] \] (8.1)
where $\mathbf{p}(\cdot)$ is a $(3 \times 1)$ a continuous vectorial function, which describes the curve when the independent variable $u$ ranges over some interval of the domain space.

In many cases, the definition of a trajectory in the task space of a robot or of a multi-axis automatic machine requires also to assign the orientation of the tool in each point of the curve. This can be achieved by specifying the configuration of the frame linked to the end effector (the tool frame) with respect to the base (world) frame. Therefore, in the general case the parametric description of the trajectory (8.1) is a six dimensional function\(^1\) providing, for each value of the variable $u$, both the position and the orientation of the tool:

$$\mathbf{p} = [x, y, z, \alpha, \beta, \gamma]^T.$$  

Therefore, the planning of a trajectory in the workspace consists in defining:

1. The function $\mathbf{p}(u)$, which interpolates a set of desired points/configurations.
2. The motion law $u = u(t)$ describing how the tool should move along the path.

In case the multi-dimensional problem can be decomposed in its components, the 3D trajectory planning can be considered as a set of scalar problems, and the techniques reported in the previous chapters can be adopted for its solution. In this case, each function $p_i(\cdot)$ depends directly on the time $t$, and the synchronization among the different components is performed by imposing interpolation conditions at the same time instants.

In this chapter, the multi-dimensional problem is approached by considering the computation of the 3D geometric path to be tracked with a prescribed motion law.

Once the trajectory in the task space has been defined, it is necessary to translate it in the joint/motor space by means of the inverse kinematic model of the system. The reader interested to this topics should refer to the specialized literature, see for example [12] and the many other excellent textbooks [79, 80, 81].

Before discussing the techniques that can be used to define the function $\mathbf{p}(u)$, some preliminary considerations and definitions are necessary. Quite often, the trajectories for the position and for the orientation are defined separately, since it can be desirable to track a well defined path in the work space, e.g. a straight line, with the orientation of the tool specified only at the endpoints of the motion. In fact, with the exception of some applications (e.g. welding, painting, etc.), not always a strict relation between position and orientation exists.

Although the two problems can be treated separately, they result conceptually

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\(^1\) In case that a minimal representation of the orientation is assumed, e.g. Euler or Roll-Pitch-Yaw angles, only three parameters are necessary, see Appendix C.