Chapter II
NONHOLONOMIC SYSTEMS

From the analog of Newton’s law, Maggi’s equations are deduced which are the most convenient equations of the nonholonomic mechanics. From Maggi’s equations the most useful forms of equations of motion of nonholonomic systems are obtained. The connection between Maggi’s equations and the Suslov–Jourdain principle is considered. The notion of ideal nonholonomic constraints is discussed. In studying nonholonomic systems the approach, applied in Chapter I to analysis of the motion of holonomic systems, is employed. The role of of Chetaev’s type constraints for the development of nonholonomic mechanics is considered. For the solution of a number of nonholonomic problems, the different methods are applied.

§ 1. Nonholonomic constraint reaction

Consider the Cartesian coordinates $Ox_1x_2x_3$ with the unit vectors $i_1$, $i_2$, $i_3$. If on the motion of material point of the mass $m$ it is imposed the nonholonomic constraint

$$\varphi(t, x, \dot{x}) = 0, \quad x = (x_1, x_2, x_3), \quad (1.1)$$

then the second Newton’s law can be represented as

$$m\ddot{w} = F + R', \quad (1.2)$$

where $F = (X_1, X_2, X_3)$ is an active force, acting on the point, and $R' = (R'_1, R'_2, R'_3)$ is constraint reaction (1.1).

Consider the vector $R'$. We differentiate equation of constraint (1.1) with respect to time:

$$\dot{\varphi} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x_k} \dot{x}_k + \frac{\partial \varphi}{\partial \dot{x}_k} \ddot{x}_k = 0, \quad k = 1, 2, 3. \quad (1.3)$$

Together with the usual vector $\nabla \varphi = \frac{\partial \varphi}{\partial x_k} i_k$ we introduce the new vector $\nabla' \varphi$ proposed by N.N. Polyakhov [185]:

$$\nabla' \varphi = \frac{\partial \varphi}{\partial \dot{x}_k} i_k.$$

Then equation (1.3) can be rewritten as

$$\frac{\partial \varphi}{\partial t} + \nabla \varphi \cdot \dot{v} + \nabla' \varphi \cdot \dot{w} = 0. \quad (1.4)$$
Multiplying scalarly equation (1.2) by $\nabla'\varphi$ and equation (1.4) by $m$, we obtain
$$
R' \cdot \nabla'\varphi = -m\left(\frac{\partial \varphi}{\partial t} + \nabla \varphi \cdot v\right) - F \cdot \nabla' \varphi.
$$
This implies that the vector $R'$ can be represented in the form
$$
R' = \Lambda \nabla' \varphi + T_0 = N + T_0,
$$
where
$$
\Lambda = -\frac{m \frac{\partial \varphi}{\partial t} + m \nabla \varphi \cdot v + F \cdot \nabla' \varphi}{|\nabla' \varphi|^2}, \quad T_0 \cdot N = 0. \tag{1.5}
$$
Note that the only component $N$ of constraint reaction depends on (1.1), in which case by formulas (1.5) it is defined as a certain function of $t, x, \dot{x}$. In particular, equations (1.1) and (1.2) are also valid for $T_0 = 0$. The nonholonomic constraints of such type we shall called ideal. If $T_0 \neq 0$, then the construction of the vector $T_0$ should be described separately, based on the additional characteristics of the physical realization of constraint (1.1). As a rule, $T_0$ essentially depends on the quantities $|N|$ and, in lesser degree, on $t, x, \dot{x}$.

Consider the partial case of holonomic constraint, namely
$$
f(t, x) = 0. \tag{1.6}
$$
Represent it in the form of (1.1):
$$
\varphi \equiv \dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_k} \dot{x}_k = 0.
$$
Then we have
$$
\frac{\partial \varphi}{\partial \dot{x}_k} = \frac{\partial f}{\partial x_k},
$$
and therefore for holonomic constraint (1.6) the vector $\nabla' \varphi$, introduced above, coincides with the usual vector $\nabla f$. Here, as is shown in Chapter I, the vector $N$ is directed along a normal to the surface, given by equation (1.6), and the vector $T_0$ lies in the plane tangential to this surface. In particular, if equation (1.6) gives a certain material surface, on which the point must move, then for the ideally burnished surface (for ideal holonomic constraint) we have $T_0 = 0$. Otherwise we need to point out a rule for construction of the vector $T_0$, for example, to give Coulomb’s law (1.12) from Chapter I.

Assume now that on the motion of material point it is imposed two nonholonomic constraints
$$
\varphi^\kappa(t, x, \dot{x}) = 0, \quad x = (x_1, x_2, x_3), \quad \kappa = 1, 2.
$$
Arguing as above, we obtain
$$
\frac{\partial \varphi^\kappa}{\partial t} + \nabla \varphi^\kappa \cdot v + \nabla' \varphi^\kappa \cdot w = 0, \quad \kappa = 1, 2.
$$