Ensuring Generality in Euclid’s Diagrammatic Arguments

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Abstract. This paper presents and compares FG and Eu, two recent formalizations of Euclid’s diagrammatic arguments in the Elements. The analysis of FG, developed by the mathematician Nathaniel Miller, and that of Eu, developed by the author, both exploit the fact that Euclid’s diagrammatic inferences depend only on the topology of the diagram. In both systems, the symbols playing the role of Euclid’s diagrams are discrete objects individuated in proofs by their topology. The key difference between FG and Eu lies in the way that a derivation is ensured to have the generality of Euclid’s results. Carrying out one of Euclid’s constructions on an individual diagram can produce topological relations which are not shared by all diagrams so constructed. FG meets this difficulty by an enumeration of cases with every construction step. Eu, on the other hand, specifies a procedure for interpreting a constructed diagram in terms of the way it was constructed. After describing both approaches, the paper discusses the theoretical significance of their differences. There is in Eu a context dependence to diagram use, which enables one to bypass the (sometimes very long) case analyses required by FG.

Each of the arguments in Euclid’s Elements, as given in [1], comes equipped with a geometric diagram. The role of the diagram in the text is not merely to illustrate the geometric configuration being discussed. It also furnishes a basis for inference. For some of Euclid’s steps, the logical form of the preceding sentences is not enough to ground the step. One must consult the diagram to understand what justifies it. Consequently, Euclid is standardly taken to have failed in his efforts to provide exact, fully explicit mathematical proofs. Inspection of geometric diagrams is thought to be too vague and open-ended a process to play any part in rigorous mathematical reasoning.

This assumption has recently been disproved. FG, developed by Nathaniel Miller and presented in [3], and Eu, developed by the author in [4], are formal systems of proof which possess a symbol type for geometric diagrams. Working within each system, one can reconstruct Euclid’s proofs in an exact and fully explicit manner, with diagrams. In this paper I compare the two systems as accounts of Euclid’s diagrammatic reasoning. In the first part, I discuss the feature of Euclid’s diagram use which makes both formalizations possible. In the second, I explain how both systems codify this use in their rules. In FG,
the content of single diagram is context independent, and the result of a geometric construction is a disjunctive array of such diagrams. Alternatively, in Euclid, the content of a diagram depends systematically on the way it is constructed in a proof. The need for the disjunctive arrays of FG, which can become very large, is thus avoided.

1 What a Diagram Can Do for Euclid

A close reading of the Elements reveals that the significance of a diagram in a proof is neither vague nor open-ended for Euclid. The first to discover this is Ken Manders, who laid out his insights on ancient geometric proof in [2].

To explain the division of labor between text and diagram in ancient geometry, Manders distinguishes between the exact and co-exact properties of geometric diagrams. Any one of Euclid’s diagrams contains a collection of spatially related magnitudes—e.g. lengths, angles, areas. For any two magnitudes of the same type, one will be greater than another, or they will be equal. These relations comprise the exact properties of the diagram. How these magnitudes relate topologically to one another—i.e. the regions they define, the containment relations between these regions—comprise the diagram’s co-exact properties. Diagrams of a single triangle, for instance, vary with respect to their exact properties. That is, the lengths of the sides, the size of the angles, the area enclosed, vary. Yet with respect to their co-exact properties the diagrams are all the same. Each consists of three bounded linear regions, which together define an area.

The key observation is that Euclid’s diagrams contribute to proofs only through their co-exact properties. Euclid never infers an exact property from a diagram unless it follows directly from a co-exact property. Exact relations between magnitudes which are not exhibited as a containment are either assumed from the outset or are proved via a chain of inferences in the text. It is not difficult to hypothesize why Euclid would have restricted himself in such a way. Any proof, diagrammatic or otherwise, ought to be reproducible. Generating the symbols which comprise it ought to be straightforward and unproblematic. Yet there seems to be room for doubt whether one has succeed in constructing a diagram according to its exact specifications perfectly. The compass may have slipped slightly, or the ruler may have taken a tiny nudge. In constraining himself to the co-exact properties of diagrams, Euclid is constraining himself to those properties stable under such perturbations.

For an illustration of the interplay between text and diagram, consider proposition 35 of book I. It asserts that any two parallelograms which are bounded by the same parallel lines and share the same base have the same area. Euclid’s proof proceeds as follows.

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1 The paper was written in 1995 but published only recently, in Philosophy of Mathematical Practice, edited by Paolo Mancosu (Clarendon Press, 2008). Despite the fact the paper existed only as draft for most of its 13 years, it has been influential to those interested in diagrams, geometry and proof. Mancosu describes it an ‘underground classic.’