Chapter 6
Multiscale Expansion and Integrability of Dispersive Wave Equations

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6.1 Introduction

The propagation of nonlinear dispersive waves is of great interest and relevance in a variety of physical situations for which model equations, as infinite-dimensional dynamical systems, have been investigated from various perspectives and to different purposes. In the ideal case in which waves propagate in a one-dimensional medium (no diffraction) without losses and sources, quite a number of special models, so-called integrable models, have been discovered together with the mathematical tools to investigate them. This important progress has provided important contributions to such matters as dispersionless propagation (solitons), wave collisions, wave decay, long-time asymptotics among others. On the mathematical side, such progress on integrable models has considerably contributed also to our present (admittedly not concise) answer to the question “What is integrability?”, which can be found in [1], and a partial guide to the vast literature on the theory of solitons is given in [2].

It is plain that integrable models, though both useful and fascinating, remain exceptional: nonlinear partial differential equations (PDEs) in 1+1 variables (space+time) are generically not integrable. The aim of these notes is to show how an algorithmic technique, based on multiscale analysis and perturbation theory, may be devised as a tool to establish how “far” is a given PDE from being integrable. This method basically associates to a given PDE one or more, generally simpler, PDEs with respect to rescaled space and time variables. This approach [3] has been known in applicative contexts [4–8] since several decades as it provides approximate solutions when only one, or a few, monochromatic “carrier waves” propagate in a strongly dispersive and weakly nonlinear medium. More recently [9] it has proved to be also a simple way to obtain necessary conditions which a given PDE has to satisfy in order to be integrable, and to discover integrable PDEs as well [10].

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The basic philosophy of this approach is to derive from a nonlinear PDE one or more PDEs whose integrability properties are either already known or easily found. In this respect, a general remark on this method is the following. Integrability is not a precise notion, and different degrees of integrability can be attributed to a PDE within a certain class of solutions and boundary conditions, according to the technique of solving it. For instance, C-integrable are those nonlinear equations which can be transformed into linear equations via a change of variables \[10\], and S-integrable are those equations which can be linearized (within a certain class of solutions) by the method of the spectral (or scattering) transform (see, f.i., \[11, 12\]).

Examples of C-integrability are the equations
\[
\begin{align*}
  u_t + a_1 u_x - a_3 u_{xxx} &= a_3 \left( 3uu_x + u^3 \right)_x, & u &= u(x,t), \\
  u_t + a_1 u_x - a_3 u_{xxx} &= 3a_3c \left( u^2 u_{xx} + 3uu_x^2 \right) + 3a_3c^2u^4u_x, & u &= u(x,t),
\end{align*}
\]
which are both mapped to their linearized version (\(a_1, a_3, c\) are constant coefficients)
\[
\begin{align*}
  v_t + a_1 v_x - a_3 v_{xxx} &= 0, & v &= v(x,t),
\end{align*}
\]
the first one, (6.1a), by the (Cole–Hopf) transformation
\[
u = v_x/v\]
and the second one, (6.2a), by the transformation \[10\]
\[
u = v/(1 + 2cw)^{1/2}, \quad w_x = v^2.
\]

Well-known examples of S-integrable equations are the modified Korteweg–de Vries (mKdV) equation
\[
\begin{align*}
  u_t + a_1 u_x - a_3 u_{xxx} &= 6a_3cu^2u_x, & u &= u(x,t), \\
\end{align*}
\]
and the nonlinear Schrödinger (NLS) equation (\(a_1, a_2, a_3, c\) are real constant coefficients)
\[
\begin{align*}
  u_t - ia_2 u_{xx} &= 2ia_2c|u|^2u, & u &= u(x,t),
\end{align*}
\]
whose method of solution is based on the eigenvalue problem
\[
\begin{align*}
  \psi_x + ik\sigma\psi &= Q\psi, & \psi &= \psi(x,k,t),
\end{align*}
\]
where \(\psi\) is a 2-dim vector, \(\sigma\) is the diagonal matrix \(\text{diag}(1,-1)\) and \(Q(x,t)\) is the off-diagonal matrix
\[
Q = \begin{pmatrix} 0 & u \\ -cu & 0 \end{pmatrix},
\]
where \(u\) is real for the mKdV equation (6.4a) and (the asterisk indicates complex conjugation)