Weak Convergence of Point Processes

One of the fundamental results of probability theory is Poisson’s limit theorem. It tells us that a sequence of binomial $\text{Bin}(p_n, n)$ distributed random variables $B_n, n = 1, 2, \ldots,$ converges in distribution to a Poisson random variable $Y$ with parameter $\lambda > 0$ if and only if $np_n = E B_n \to EY = \lambda.$ Here one deals with a rare event approximation because $p_n$, the success probability of the binomial random variable $B_n$, necessarily converges to zero. It is a remarkable result insofar that the distributional convergence relation $B_n \xrightarrow{d} Y$ is equivalent to the convergence of the expectations $E B_n \to EY$. We will see later that this property remains valid in a sense for the convergence in distribution of “binomial point processes” towards a Poisson random measure (PRM): convergence of the underlying mean measures of the “binomial point processes” implies their convergence in distribution towards a PRM with the limiting mean measure.

Any binomial $\text{Bin}(n, p)$ random variable $B_n$ can be interpreted as the counting number of the successes in $n$ independent trials with success probability $p \in (0, 1)$: if $(X_i)$ is an iid sequence which describes an experiment with the success event $A$ with probability $p$, then the number of successes

$$B_n = \sum_{i=1}^{n} I_A(X_i), \quad n \geq 1,$$

is binomially distributed with parameter $(n, p)$. This representation bears some resemblance to a point process. Indeed, we can introduce point processes

$$N_n = \sum_{i=1}^{n} \varepsilon_{X_i}, \quad n = 1, 2, \ldots,$$

generated from the points $X_1, X_2, \ldots$, on a suitable state space $E$. Then $B_n = N_n(A)$, and we can vary the event $A$ over some $\sigma$-field $\mathcal{E}$. Since we know the notion of a PRM as a special point process it is reasonable to think
about distributional convergence of the “binomial processes” \(N_n\) to a PRM. Of course, since \(EN_n(A) = n P(X_1 \in A) \to \infty\) or \(= 0\) according as \(P(X_1 \in A) > 0\) or \(= 0\) we need to ensure that \(A\) depends on \(n\) and then \(P(X_1 \in A_n) \to 0\) at a certain rate; we will often achieve this goal by considering the point process of the normalized and centered points \(c_n^{-1}(X_i - d_n)\) for suitable constants \(c_n > 0\) and \(d_n \in \mathbb{R}\).

In what follows, we will deal with two major problems. First, point processes are random measures: each such random measure can be understood as a collection of random variables indexed by the sets of an appropriate \(\sigma\)-field which typically contains infinitely many elements. We have to clarify the meaning of convergence in distribution for these infinite-dimensional objects; see Section 9.1.1. Second, we need to explain the meaning of convergence of (possibly infinite) mean measures. Weak convergence of probability measures will be a guide. It will be one of the topics in Section 9.1.2, where we introduce the concept of vague convergence of measures. There we will also see that the convergence in distribution of point processes is equivalent to the pointwise convergence of the underlying Laplace functionals.

In Section 9.2 we apply the results of Section 9.1 in the context of extreme value theory. Extremes are important for applications in non-life insurance, in particular, in reinsurance. We study the point processes of exceedances and their weak convergence to a PRM. This convergence is equivalent to the convergence of maxima and upper order statistics for iid sequences. In Section 9.2.2 we characterize the possible limit distributions for maxima of iid random variables. The celebrated Fisher-Tippett Theorem (Theorem 9.2.7) summarizes the asymptotic theory for maxima. In Section 9.2.3 we consider maximum domains of attraction, i.e., we find conditions on a distribution \(F\) such that the normalized and centered partial maxima of an iid sequence with distribution \(F\) converge to a non-degenerate limit. In Section 9.2.4 we modify the point process of exceedances insofar that we assume that we consider a random sample indexed by a renewal process. The weak convergence of these processes is applied to reinsurance treaties of extreme value type as introduced in Section 3.4. In Section 9.3 we return to these treaties and study their limiting behavior.

9.1 Definition and Basic Examples

9.1.1 Convergence of the Finite-Dimensional Distributions

First of all we have to clarify:

What is the meaning of weak convergence of point processes?

This question cannot be answered at a completely elementary level. Consider point processes \(N, N_1, N_2, \ldots\) on the same state space \(E \subset \mathbb{R}^d\). We know from Section 7.1.2 that the distribution of these point processes in \(M_p(E)\), the space