Chapter 4
Principle of Least Action in Continuum Mechanics

As we discussed in Sect. 2.6, for reversible processes the governing equations of mechanics must have a Hamiltonian structure, and accordingly a principle of least action must exist. In another extreme case, when the inertial effects and the internal interactions described by the internal energy can be ignored, variational principles also exist, but they are due to the special structure of the models rather than to the laws of Nature. In this chapter we consider the principle of least action in continuum mechanics of reversible processes and some related issues. The variational principles for dissipative processes are presented in the second part of the book along with the other variational features of the classical continuum models.

4.1 Variation of Integral Functionals

Let a continuum be characterized by some functions $u^\kappa(x^i,t)$, $\kappa = 1, \ldots, m$, $i = 1, 2, 3$. In what follows the number of the variables $x^i$ is not essential; besides, time does not play a special role. Therefore, we include time in the set of independent variables and write $u^\kappa = u^\kappa(x)$ assuming that $x = \{x^1, \ldots, x^n\}$ is a point of $n$-dimensional space. We write, for brevity, $u(x)$ for the set $\{u^1(x), \ldots, u^m(x)\}$, when this cannot cause confusion.

Let a functional, $I(u)$, be given, i.e. there is a rule which allows one to compute the number, $I(u)$, for each $u(x)$. The major example of such a functional for us is an integral functional

$$I(u) = \int_V L(x^i, u^\kappa, u_i^\kappa)dx^i, \quad u_i^\kappa \equiv \frac{\partial u^\kappa}{\partial x^i},$$

(4.1)

where $V$ is some region in $n$-dimensional space of $x$-variables. For such functionals, function $L$ is called Lagrangian. The case when Lagrangian depends on higher derivatives will be considered further in Sect. 4.4.

We assume that region $V$ is compact to avoid the technicalities caused by the unboundedness of $V$; some peculiarities of the variational problems for an unbounded domain are considered in Sect. 11.7. All functions involved are assumed
to be sufficiently smooth to make the derivation of the final equations meaningful. In particular, the boundary \( \partial V \) of region \( V \) is piecewise smooth, and Lagrangian is a smooth function of its arguments.

Physical reasoning to be considered later sets some constraints on the admissible function \( u(x) \). A typical constraint is prescribing the values of \( u(x) \) at a piece of the boundary, \( \partial V_u \), of the boundary \( \partial V \):

\[
    u(x) = u_b(x) \quad \text{at} \quad \partial V_u. \tag{4.2}
\]

Here \( u_b(x) \) are the given boundary values of functions \( u(x) \).

The constraints specify the set of admissible functions \( u(x) \). We denote this set by \( M \). A description of the set \( M \) also includes the characterization of smoothness of the admissible functions. In order for the integral (4.1) to be sensible, it is sufficient to include in the set \( M \) the functions \( u(x) \) which are continuous along with their derivatives in the closed region \( V \). The physically important case of piecewise differentiable functions will be treated in Sect. 7.4.

A variational principle usually states that the true process (or, in statics, the equilibrium state) of a continuum is a stationary point of the functional \( I(u) \) on the set \( M \), i.e. variation of this functional, \( \delta I \), vanishes for all admissible variations \( \delta u \).

Let \( u(x) \) be a stationary point of the functional \( I(u) \) and \( u(x) = u(x) + \delta u(x) \) be a small disturbance of the function \( u(x) \). Keeping only the leading small terms in the difference \( I(u + \delta u) - I(u) \), we find the variation of the functional, \( I(u) \),

\[
    \delta I = \int_V \left( \frac{\partial L}{\partial u^\kappa} \delta u^\kappa + \frac{\partial L}{\partial (\delta u^\kappa)_i} (\delta u^\kappa)_i \right) d^n x. \tag{4.3}
\]

Functions \( \delta u^\kappa \) and \( (\delta u^\kappa)_i \) are not independent because \( (\delta u^\kappa)_i \) are completely determined by \( \delta u^\kappa \). To put (4.3) in the form which contains only independent terms, we integrate the second term in the integrand by parts:

\[
    \delta I = \int_V \left( \left( \frac{\partial L}{\partial u^\kappa} - \frac{\partial}{\partial x^i} \frac{\partial L}{\partial (\delta u^\kappa)_i} \right) \delta u^\kappa + \frac{\partial}{\partial x^i} \left( \frac{\partial L}{\partial (\delta u^\kappa)_i} \delta u^\kappa \right) \right) d^n x \\
    = \int_V \frac{\partial L}{\partial u^\kappa} \delta u^\kappa d^n x + \int_{\partial V} \frac{\partial L}{\partial (\delta u^\kappa)_i} \delta u^\kappa n_i dA. \tag{4.4}
\]

Here \( dA \) is the area element at \( \partial V \), \( n_i \) the unit normal vector at \( \partial V \) and a notation for the variational derivative of \( L \) is introduced,

\[
    \frac{\delta L}{\delta u^\kappa} = \frac{\partial L}{\partial u^\kappa} - \frac{\partial}{\partial x^i} \frac{\partial L}{\partial (\delta u^\kappa)_i}. \tag{4.5}
\]

Note that

\[
    \delta u = 0 \quad \text{at} \quad \partial V_u
\]