Chapter 8
Dynamics of Elastic Bodies

8.1 Least Action vs Stationary Action

The extrapolation to dynamics of the minimization principles formulated above encounters difficulties, the essence of which can be observed for systems with one degree of freedom.

Consider the harmonic oscillator – a material point on a spring. The deviation of the point from the equilibrium position is denoted by \( x (t) \); the kinetic energy is equal to \( \frac{1}{2} m \dot{x}^2 \), and the energy of the spring is \( \frac{1}{2} k x^2 \). According to the Hamilton principle, the true trajectory is the stationary point of the functional

\[
I = \frac{1}{2} \int_0^{\Delta t} (m \dot{x}^2 - kx^2) dt
\]

on the set of functions, \( x(t) \), which take at the initial and final moments the given values

\[
x (0) = x_0, \quad x (\Delta t) = x_1.
\]

The question is: does the true trajectory provide the minimum for the functional \( I \)? In order to investigate this question, it is convenient to use instead of functions \( x(t) \) the functions \( u(t) \) which are equal to zero in the initial and final moments:

\[
x \rightarrow u : x (t) = vt + x_0 + u (t), \quad v = (x_2 - x_1) / \Delta t = \text{const}, \quad u (0) = u (\Delta t) = 0.
\]

The constant, \( v \), has the meaning of the average velocity. The functional \( I (u) \) takes the form (additive constant is omitted)

\[
I (u) = J (u) + l (u),
\]

\[
J (u) = \frac{1}{2} \int_0^{\Delta t} (m \dot{u}^2 - ku^2) \, dt, \quad l (u) = -k \int_0^{\Delta t} (vt + x_o) u \, dt.
\]
Posing the minimization problem is possible if the functional $I(u)$ is bounded from below. For this to be true, it is necessary that the functional $J(u)$ be bounded from below. For every function $u(x)$, the set of admissible functions contains the function $\lambda u(x)$ (with any $\lambda$); therefore, for the quadratic functional to be bounded from below it is necessary and sufficient that it is non-negative. Let us write down this condition. It is convenient to use a new argument $\tau = \frac{2\pi}{\Delta t} t$. From the inequality $J(u) \geq 0$, it follows that, for any functions $u(\tau)$ such that $u(0) = u(2\pi) = 0$, the inequality

$$
\int_0^{2\pi} \left(\frac{du}{d\tau}\right)^2 d\tau \geq c \int_0^{2\pi} u^2 d\tau, \quad c = \frac{k(\Delta t)^2}{4\pi^2 m}
$$

(8.1)

should hold. We arrive at the Wirtinger inequality (5.24). The inequality (8.1) holds for all $c \leq 1/4$. Moreover, for $c < 1/4$ the functional $J(u)$ will be strictly convex, as a quadratic positive functional. For $c > 1/4$, the functional $J(u)$ is not bounded from below. Indeed, $1/4$ is the best constant in the Wirtinger inequality for zero values of the function at the ends; therefore, for $c > 1/4$ there exists at least one function $u_0$ for which $J(u_0) < 0$, while $l(u_0)$ has a finite value. Thus, for the sequence $\{\lambda u_0\}, \lambda \to \infty$, $J(\lambda u_0) \to -\infty$.

So, posing the minimization problem is possible only for sufficiently small $\Delta t$, $\Delta t < \pi \sqrt{m/k}$.

For continuous media, the problem is more complicated. Let us consider, for example, the action functional of the wave equation

$$
I(u) = \frac{1}{2} \int_0^\Delta t \int_{-\pi}^\pi \left(u_{,t}^2 - u_{,x}^2\right) dx dt
$$

and set the kinematic boundary conditions $u(0, x) = u_0(x), u(\Delta t, x) = u_1(x), u(t, -\pi) = u(t, \pi) = 0$. Let the functions $u_0(x)$ and $u_1(x)$ be odd, so that we will only need odd admissible functions $u(x)$. Functions $u(x)$ can be presented in the form of the Fourier series:

$$
u(t, x) = \sum_{k=1}^\infty u_k(t) \sin kx.
$$

(8.2)

Substituting (8.2) into $I(u)$, we get

$$
I(u) = \sum_{k=1}^\infty I_k, \quad I_k = \frac{\pi}{2} \int_0^{\Delta t} \left(\ddot{u}_k - k^2 u_k^2\right) dt.
$$

---

1 Indeed, if there exists a sequence $\{u_n\}$ for which $J(u_n) \to -\infty$, while $l(u_n) \leq \text{const}$, then $I(u_n) \to -\infty$. If $l(u_n) \to +\infty$, then for the sequence $\{-u_n\}$, we have $I(u_n) \to -\infty$.

2 See Sect. 5.13.