Chapter 10
Self-Similar Nanostructures

The geometric description of the world is at the root of Western thought: to the first (perhaps mythic) philosopher, Thales, we owe a theorem of plane geometry. The role of geometry in the construction of physical theories is equally important: on the cosmological scale, gravitation is explained in terms of space curvature; on the submicroscopic scale, quantum mechanics may be explained in terms of fluctuation of the metric tensor [303].

On the laboratory length scale, the most important geometric property of any thermodynamic system is most likely its shape. The general mathematical description of surfaces is a notoriously complicated affair – especially if one tries to describe real surfaces.

10.1 Fractals

Fractals are self-similar objects. Pathological objects with self-similar characteristics were already known to mathematicians at the end of the nineteenth century (Pean and Koch curves, Cantor middle-excluded set, etc.); however, fractals took on a central role in the description of the physical system much later, with Mandelbrot’s observation of the scale invariance of several physicochemical systems [304].

10.1.1 Queer Systems

The area \( A \) of bodies with a regular shape (such as the cube, sphere, polyhedra, etc.) increases with volume as \( V^{2/3} \). For bodies with a regular shape, the effect of the surface on their intensive thermodynamic properties (such as specific heat, magnetic susceptibility, etc.) disappears in the thermodynamic limit (infinite volume at constant density).

A body whose area increases with \( V \) faster than \( V^{2/3} \) is said to have a queer shape. The area of a queer system is well defined; its effect on intensive thermodynamic properties, however, can persist even for \( V \to \infty \). The effect surely persists in the thermodynamic limit for bodies with \( A \propto V \).
Though it may appear that the concept of a queer system is an extravagant concept, Nature abounds in bodies with queer shape (the first systematic analysis for queer systems was recorded in [305]). Among the most interesting queer systems are zeolites (in which queerness is due to a lattice of void cages connected by tubes, regularly arranged in the system), biological systems (because reproduction is a way which allows the overall area to increase in proportion to the volume), and films obtained by low-temperature PVD (in which the condensed film grows with an exposed number of sites increasing in proportion to the average thickness [306]).

Queer systems, however, are not sufficiently extravagant to exhaust the possibilities of Nature – fractals are even more extravagant.

10.1.2 Fractals in Mathematics

That smoothness is not a mandatory feature of the way Nature expresses itself, but rather, fractality is of ubiquitous occurrence in a large class of phenomena even in the Mineral Kingdom, has been clear ever since the genesis of the theory, with Mandelbrot’s question about the length of Great Britain’s coast [304].

From a mathematical point of view, a fractal set exhibits the property that the “whole” can be represented as the collection of several parts, each one obtainable from the “whole” by a contracting similitude [304]. A typical fractal object is self-similar, i.e., a magnified portion of it appears identical to the entire object observed under lower resolution: from this point of view it is said to be invariant under scale transformation. If the contraction similitude occurs in less than three dimensions, the object is said to be self-affine.

A fractal object can usually be defined through a successive iteration process in which an initiator is contracted with a similarity ratio \( \chi \) and put \( v(\chi) \) times in a given arrangement called generator, the same operation then being repeated again and again without end. If the area of a fractal set varies with the probe yardstick \( \rho \) as in

\[
A \propto \rho^{2-D},
\]

(10.1)

(which diverges for \( D > 2 \)), the fractal dimension \( D \) of the set is given by\(^1\)

\[
D = - \lim_{\chi \to \infty} \left| \frac{\ln v(\chi)}{\ln(1/\chi)} \right|.
\]

\(^1\) For any set \( X \) in \( \mathbb{R}^n \) its dimension, \( \text{dim}(X) \), is a positive number with domain \([0, n]\) satisfying the following properties [307]:

- (Limiting behaviors) for the singlet set \( \{P\} \), \( \text{dim} \{P\} = 0 \); for the interval \( I \), \( \text{dim} I = 1 \); for the hypercube \( I^m \ (m \leq n) \), \( \text{dim} I^m = m \)
- (Monotonicity) \( X \subseteq Y \implies \text{dim}(X) \leq \text{dim}(Y) \)