In this chapter we add to an affine space $X$ the set $P(X)$ of directions of its lines, obtaining a projective space in which $X$ is naturally embedded. Conversely, the complement of a hyperplane in a projective space has a natural affine structure. This process of projective completion can be performed in more or less natural ways (section 5.0). In section 5.3 we translate parallelism behavior in $X$ in terms of intersection properties in the completion of $X$. This has numerous applications, some of which are given in section 5.4.
5.0. Introduction

5.0.1. We have mentioned in 4.0 the shortcomings of affine geometry. In particular, following Desargues, we want to extend an affine space $X$ into a projective space $\tilde{X}$, the union of $X$ and its points at infinity. These are defined as the directions of lines of $X$, and their set $P(\tilde{X})$ is also written $\infty_X$. Thus $\tilde{X} = X \cup \infty_X = X \cup P(\tilde{X})$, this being a priori just a disjoint union in the set-theoretical sense. The non-trivial part is making $\tilde{X}$ into a projective space. This can be done axiomatically, but this is not the point of view we have been adopting in this book (cf. 4.1.2 and [AN], [HA], [DI], [H-P], [PT]); instead we present three algebraic constructions for the projective completion.

5.0.2. In the coarsest approach, let $X$ have finite dimension $n$, and let $\{x_i\}_{i=0}^n$ be an affine frame of $X$. The desired completion is

$$P^n(K) = P(K^{n+1}),$$

and the embedding $X \rightarrow P^n(K)$ is given by

5.0.2.1

$$x = (\lambda_1, \ldots, \lambda_n) \mapsto p(\lambda_1, \ldots, \lambda_n, 1)$$

(cf. 2.2.9 and 4.2). We see that the complement of the image of $X$ in $P^n(K)$ is exactly the hyperplane $P^{n-1}(K) = P(K^n \cong K^n \times \{0\})$, which is indeed identified with $P(\tilde{X})$ by 5.0.2.1.

5.0.3. A more sophisticated construction, suggested by 5.0.2, consists in vectorializing $X$ at $a \in X$, and considering the projective space $P(X_a \times K)$, where $X_a \times K$ is the direct product of the two vector spaces $X_a$ and $K$. The embedding of $X$ into $P(X_a \times K)$ is furnished by composing the identification between $X$ and the section $X_a \times \{1\}$ of $X_a \times K$ with the projection onto the quotient:

5.0.3.1

$$x \mapsto p(x, 1).$$

The complement of the image of $X$ in $P(X_a \times K)$ is $P(X_a) \cong P(\tilde{X})$.