On the OBDD Complexity of Threshold Functions and the Variable Ordering Problem
(Extended Abstract)

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Abstract. Ordered binary decision diagrams (OBDDs) are one of the most common dynamic data structures for Boolean functions. Among the many areas of application are verification, model checking, computer aided design, relational algebra, and symbolic graph algorithms. Threshold functions are the basic functions for discrete neural networks and are used as building blocks in the design of symbolic graph algorithms. In this paper the first exponential lower bound on the size of a more general model than OBDDs and the first nontrivial asymptotically optimal bound on the OBDD size for a threshold function are presented. Furthermore, if the number of different weights is a constant it is shown that computing an optimal variable order for multiple output threshold functions is NP-hard whereas for single output function the problem is solvable in deterministic polynomial time.

Keywords: Computational complexity, ordered binary decision diagrams, threshold functions, variable ordering problem.

1 Introduction and Results

1.1 Ordered Binary Decision Diagrams and Threshold Functions

When working with Boolean functions as in circuit verification, synthesis, and model checking, ordered binary decision diagrams, denoted OBDDs, introduced by Bryant [6], are one of the most often used data structures supporting all fundamental operations on Boolean functions. Furthermore, in the last years a research branch has emerged which is concerned with the theoretical design and analysis of so-called symbolic algorithms which solve graph problems on OBDD-represented graph instances (see, e.g., [9,10], [23]).

Definition 1. Let $X_n = \{x_1, \ldots, x_n\}$ be a set of Boolean variables. A variable order $\pi$ on $X_n$ is a permutation on $\{1, \ldots, n\}$ leading to the ordered list $x_{\pi(1)}, \ldots, x_{\pi(n)}$ of the variables.

In the following a variable order $\pi$ is sometimes identified with the corresponding order $x_{\pi(1)}, \ldots, x_{\pi(n)}$ of the variables if the meaning in clear from the context.
Definition 2. A $\pi$-OBDD on $X_n$ is a directed acyclic graph $G = (V, E)$ whose sinks are labeled by Boolean constants and whose non sink (or inner) nodes are labeled by Boolean variables from $X_n$. Each inner node has two outgoing edges one labeled by 0 and the other by 1. The edges between inner nodes have to respect the variable order $\pi$, i.e., if an edge leads from an $x_i$-node to an $x_j$-node, $\pi^{-1}(i) \leq \pi^{-1}(j)$ ($x_i$ precedes $x_j$ in $x_{\pi(1)}, \ldots, x_{\pi(n)}$). Each node $v$ represents a Boolean function $f_v : \{0,1\}^n \rightarrow \{0,1\}$ defined in the following way. In order to evaluate $f_v(b)$, $b \in \{0,1\}^n$, start at $v$. After reaching an $x_i$-node choose the outgoing edge with label $b_i$ until a sink is reached. The label of this sink defines $f_v(b)$. The size of a $\pi$-OBDD $G$ is equal to the number of its nodes and the $\pi$-OBDD size of a function $f$, denoted by $\pi$-OBDD($f$), is the size of the minimal $\pi$-OBDD representing $f$.

SBDDs (shared binary decision diagrams) are an extension of OBDDs that can express multiple functions. An SBDD represents a Boolean function $f \in B_{n,m} : \{0,1\}^n \rightarrow \{0,1\}^m$ by representing simultaneously the output functions $f_1, f_2, \ldots, f_m$ of $f$, where the representations for the different coordinate functions $f_1, f_2, \ldots, f_m$ may share nodes.

The size of the the reduced $\pi$-SBDD representing $f$ is described by the following structure theorem [13].

Theorem 1. The number of $x_{\pi(i)}$-nodes of the $\pi$-SBDD for $f = (f_1, \ldots, f_m)$ is the number $s_i$ of different subfunctions $f_j|x_{\pi(i)}=a_1, \ldots, x_{\pi(i-1)}=a_{i-1}$, $1 \leq j \leq m$ and $a_1, \ldots, a_{i-1} \in \{0,1\}$, essentially depending on $x_{\pi(i)}$ (a function $g$ depends essentially on a Boolean variable $z$ if $g|_{z=0} \neq g|_{z=1}$).

It is well known that the size of an OBDD or an SBDD representing a function $f$ depends on the chosen variable order and may vary between linear and exponential size. Since in applications the variable order $\pi$ is not given in advance we have the freedom (and the problem) to choose a good order for the representation of $f$.

Definition 3. The OBDD size or OBDD complexity of $f$ (denoted by OBDD($f$)) is the minimum of all $\pi$-OBDD($f$). Analogously the SBDD size or SBDD complexity of a multiple output function is defined.

A variable order $\pi$ is called optimal for a Boolean function $f$ if $\pi$-OBDD($f$) ($\pi$-SBDD($f$)) is equal to OBDD($f$) (SBDD($f$)).

Read-once branching programs, denoted BP1s, also sometimes called free binary decision diagrams, are a more general model where the variable orders on different paths may be different.

Threshold functions are the basic functions for discrete neural networks.

Definition 4. The threshold function $T_{w_1, \ldots, w_n, t} \in B_n$ with integer weights $w_1, \ldots, w_n$ and threshold value $t$ computes 1 on input $b$ iff $b_1 w_1 + \cdots + b_n w_n \geq t$. The sum $\sum_{i=1}^n b_i w_i$ is called the weight of the input $b$.

Obviously, the output of a threshold function depends on a partial input $(b_1, \ldots, b_i)$ only via the partial sum $b_1 w_1 + \cdots + b_i w_i$ and $O(n(w_1 + \cdots + w_n))$ is