Chapter 21
Theory of Errors

21.1 Purpose of the Theory of Errors

The theory of errors is a part of mathematical statistics and deals with the following facts.

Given the results of measurements carried out in a laboratory, we require statements about the ‘true’ value of the measured quantity and a prediction of the accuracy of the measurements.

There are two types of errors which arise when we carry out a measurement: 
*systematic* or *constant errors* and *random errors*.

*Constant errors* are errors generated in the measuring instruments or in the method of measurement. They always bias the result in a particular direction so that it is either too large or too small; they arise through wrong calibration of the measuring instrument or not paying attention to secondary effects. An example of a constant error is often found in the speedometer of a car, sometimes as a result of design. The speed indicated is frequently found to be 5% above the ‘true’ speed of the car, but this can vary from 0 to 7% in practice.

Constant errors can only be avoided by a critical analysis of the measuring technique and of the instruments, and such errors cannot be discovered with the help of the theory of errors.

*Random errors* are due to interference during measurements, so that a repetition of measurements does not give exactly the same results, i.e. the measured values vary. For example, if we weigh a body repeatedly we will always obtain a different result. Although we take great care on each occasion, we are not able to read each time exactly the same position of a pointer between two very fine marks. Furthermore, the pointer itself does not always settle at the same position. Random errors are the result of a multiplicity of interference factors like the fluctuation of the boundary conditions which had initially been assumed to be controllable (temperature, air pressure, voltage fluctuations, shocks and errors of observation).

To avoid random errors we must naturally improve the method of measurement. This leads to more reliable measurements but does not solve the fundamental prob-
lem. The influence of random errors can be limited, and the accuracy of measurement may even be improved in powers of 10; however, each instrument has limited accuracy and hence random errors will always appear.

The purpose of the theory of errors can now be formulated more precisely.

From the measured values we want to be able to infer the ‘true’ value of the measured quantity and estimate the reliability of the measurement. Each reading in an experiment is made up of a hypothetical ‘true’ value of the measured quantity and an error component:

\[ x = T + E \]

where \( x \) = measured value, \( T \) = ‘true’ value free of errors, and \( E \) = error component. Furthermore,

\[ E = E_1 + E_2 \]

where \( E_1 \) = random error which can be estimated by repeated measurements, and \( E_2 \) = constant error.

### 21.2 Mean Value and Variance

#### 21.2.1 Mean Value

The acceleration due to gravity, \( g \), is to be determined experimentally. The time of fall of a sphere is measured with a stopwatch and the distance with a tape measure. In order to increase the reliability, the measurements are repeated in a series of readings. A series of readings comprising 20 measurements is considered to be a random sample of all possible measurements for this experimental set-up. The arithmetic mean value of \( n \) measurements is taken as the best estimate for the ‘true’ value. We have

\[ \bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j \]

If individual measured values occur repeatedly, the mean value can be expressed in terms of the frequency \( h_i \) with which they occur, in which case we have

\[ \bar{x} = \sum_{i=1}^{k} h_i x_i \]

where \( n_i \) is the frequency and \( h_i = n_i / n \) is the relative frequency of the measured value \( x_i \). If \( n \to \infty \), the relative frequencies become the probabilities \( P(i) \) (see Chap. 19, Sect. 19.2).

The sum of all deviations from the arithmetic mean value vanishes, i.e.

\[ \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - \bar{x}) = 0 \]