Chapter 14 Analytical Trial Function Method — Membrane and Plate Bending Elements

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Abstract This chapter introduces a novel finite element method, namely, the analytical trial function method. A detailed discussion on the features of the analytical trial function method is firstly given in Sect. 14.1. Then, in the next five sections, the basic analytical solutions of plane problem, thick plate problem and thin plate problem are derived and taken as the trial functions for the corresponding finite element models. It can be seen that those resulting models exhibit excellent performance. Some challenging problems, such as the trapezoidal locking and shear locking, can be avoided naturally.

Keywords finite element, analytical trial function method, membrane element, plate bending element.

14.1 Recognition of the Analytical Trial Function Method

14.1.1 Trial Function

When constructing a displacement-based element, the first step is usually to assume its displacement mode. For example, the displacement mode of the constant strain triangular element CST is assumed to be

\[
\begin{align*}
\{u\} &= \begin{bmatrix} 1 & 0 & x & 0 & y & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_6 \end{bmatrix} = F\lambda
\end{align*}
\] (14-1)
and the displacement mode of the rectangular thin plate element ACM is assumed to be

$$w = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_{12} \end{bmatrix} = F\lambda$$ (14-2)

where $\lambda$ is the undetermined parameter vector; $F$ is composed of trial functions (or basis functions).

Element performance relies deeply on the selected trial functions. As a result of the irrationally selected trial functions for displacements, trapezoidal locking and shear locking phenomena may exist in some membrane and thick plate elements, respectively.

### 14.1.2 Analytical Trial Function

In structural matrix analysis, the exact (or analytical) solutions for the displacement of the thin beam theory are used by the thin beam element. That is to say, the selected trial functions are analytical solutions; therefore, they are called as analytical trial functions.

Timoshenko thick beam element also uses the exact solutions for the displacements of the thick beam theory. So, it will not suffer from shear locking phenomenon because the analytical trial functions are employed.

2D and 3D elasticity problems, thin and thick plate problems are all problems with infinite DOFs. For their homogeneous problems, the analytical solutions are composed of infinite terms. Finite element method is an approximate method in which such infinite DOF problems are treated as problems with only finite DOFs. And, the corresponding element model also contains finite ($n$) DOFs, which means that its displacement mode contains only $n$ basic analytical trial functions. These $n$ analytical solutions can be selected in turn from low to high orders.

### 14.1.3 Analytical Trial Function Method

The construction procedure of the finite element model in which the basic analytical solutions are taken as the trial functions is called the finite element method based on analytical trial functions, or the analytical trial function method.

The feature of the analytical trial function method is: the finite element method is a discrete approximate method, while the advantages of the analytical method are preserved in it. It exhibits the close relation between the trial function and the