In Chapters 3 and 4 we consider a variation of the SAW in which self-intersections are not forbidden but are penalized. We refer to this as the soft polymer. In Chapter 3 will show that the soft polymer has ballistic behavior in $d = 1$. The proof uses a Markovian representation of the local times of one-dimensional SRW (a powerful technique that is useful also for other models), in combination with large deviation theory, variational calculus and spectral calculus. In Chapter 4 we will show that the soft polymer has diffusive behavior in $d \geq 5$. The proof there uses a combinatorial expansion technique called the lace expansion, and is based on the idea that in high dimension SAW can be viewed as a “perturbation” of SRW.

The above scaling says that in $d = 1$ and $d \geq 5$ the soft polymer is in the same universality class as SAW. This is expected to be true also for $2 \leq d \leq 4$, but a proof is missing.

In Section 3.1 we define the model, in Section 3.2 we state the main result, a large deviation principle (LDP) for the location of the right endpoint. In Section 3.3 we outline a five-step program to prove the LDP for bridge polymers, i.e., polymers confined between their endpoints. This program is carried out in Section 3.4. In Section 3.5 we remove the bridge condition and prove the full LDP. It will turn out that the rate function has an interesting critical value strictly below the typical speed. The main technique that is used is the method of local times.

### 3.1 A Polymer with Self-repellence

The soft polymer on $\mathbb{Z}^d$ treated in Chapters 3 and 4 is defined by choosing the set of paths and the Hamiltonian in (1.1) as

$W_n = \{ w = (w_i)_{i=0}^n \in (\mathbb{Z}^d)^{n+1} : w_0 = 0, \|w_{i+1} - w_i\| = 1 \forall 0 \leq i < n \}$

$H_n(w) = \beta I_n(w), \quad (3.1)$

where $\beta \in [0, \infty)$ and

$$I_n(w) = \sum_{i,j=0}^{n} 1_{\{w_i = w_j\}}$$

is the intersection local time of $w$. This model goes under the name of weakly self-avoiding random walk: every self-intersection contributes an energy $\beta$ to the Hamiltonian and is therefore penalized by a factor $e^{-\beta}$. Another name used in the literature is Domb-Joyce model. Think of $\beta$ as a strength of self-repellence parameter.

We write $P_\beta^\beta_n$ to denote the law of the soft polymer of length $n$ with parameter $\beta$, as in (1.2). We add a factor $(1/2^d)^n$ to $P_\beta^\beta_n$ in order to be able to compare it with the law $P_n$ of SRW, i.e., we put

$$P_\beta^\beta_n(w) = \frac{1}{Z_\beta^\beta_n} e^{-\beta I_n(w)} P_n(w), \quad w \in W_n,$$

so that we may think of $P_\beta^\beta_n$ as an exponential tilting of $P_n$. Thus, $P_\beta^\beta_n$ is the law of a random process $(S_i)_{i=0}^{n}$ with weak self-repellence, taking values in $W_n$. Note that, like for SAW in Section 1.2, $(P_\beta^\beta_n)_{n \in \mathbb{N}_0}$ is not (!) a consistent family when $\beta \in (0, \infty)$. The case $\beta = 0$ corresponds to SRW, the case $\beta = \infty$ to SAW.

In what follows we focus on the case $d = 1$. In Chapter 4 we deal with the case $d \geq 5$.

### 3.2 Weakly Self-avoiding Walk in Dimension One

Intuitively, we expect that typical paths under the measure $P_\beta^\beta_n$ hang around the origin for a while and then wander off to infinity at a strictly positive speed because of the self-repellence (there is a trivial symmetry between moving to the left and moving to the right). Ballistic behavior was first shown by Bolthausen [25], without existence and identification of the speed. Theorems 3.1 and 3.2 below, which are taken from Greven and den Hollander [130], establish existence and identify the speed in terms of a variational problem. See also den Hollander [168], Chapter IX.

**Theorem 3.1.** For every $\beta \in (0, \infty)$ there exists a $\theta^*(\beta) \in (0, 1)$ such that

$$\lim_{n \to \infty} P_\beta^\beta_n\left(\left|\frac{1}{n} S_n - \theta^*(\beta)\right| \leq \epsilon \bigg| S_n \geq 0\right) = 1 \text{ for all } \epsilon > 0.$$

**Theorem 3.2.** The function $\beta \mapsto \theta^*(\beta)$ can be computed in terms of a variational problem. It follows from the solution of this variational problem that $\beta \mapsto \theta^*(\beta)$ is analytic on $(0, \infty)$,

$$\lim_{\beta \downarrow 0} \theta^*(\beta) = 0, \quad \lim_{\beta \to \infty} \theta^*(\beta) = 1.$$