Lecture 10  
Resistivity and Viscosity

Resistance is futile.  
The Borg, *Star Trek: The Next Generation*

In this lecture, we briefly discuss the effect of resistivity and viscosity on the dynamics of a magnetized fluid.

We just proved that the change in magnetic flux passing through a co-moving closed circuit is

$$\frac{d\Psi}{dt} = - \oint_C (\mathbf{E} + \mathbf{V} \times \mathbf{B}) \cdot d\mathbf{l}. \quad (10.1)$$

Since in ideal MHD $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$, we have $d\Psi/dt = 0$, and we say that the flux is “frozen in” the fluid.

However, in the more general MHD case when the fluid is no longer a perfect electrical conductor, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$ and

$$\frac{d\Psi}{dt} = - \oint_C \eta \mathbf{J} \cdot d\mathbf{l} \neq 0, \quad (10.2)$$

so that the frozen flux condition no longer applies. This is called *resistive MHD*. In this case, the fluid can “move” separately from the field and the field lines can “slip across” the fluid. We will eventually see that this can be an important effect, even when the resistivity is very small.

In resistive MHD, the combination of Faraday’s law and Ohm’s law becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right). \quad (10.3)$$

The first term is just ideal MHD. The second term is a modification introduced when the electrical conductivity $\sigma = 1/\eta$ is finite (rather than infinite). When $\eta = \text{constant}$, the last term can be written as

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\[ \nabla \times \left( \frac{\eta}{\mu_0} \nabla \times \mathbf{B} \right) = \frac{\eta}{\mu_0} \nabla \times \nabla \times \mathbf{B} \]
\[ = \frac{\eta}{\mu_0} \left[ \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \right] \]
\[ = -\frac{\eta}{\mu_0} \nabla^2 \mathbf{B}, \]

so that Eq. (10.3) becomes
\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}. \] (10.4)

The effect of resistivity is to introduce diffusion of the magnetic field, with a diffusion coefficient \( D_\eta = \eta/\mu_0 \) (m²/s). The characteristic time scale for the diffusion of structures with length scale \( L \) is
\[ \tau_R = L^2 D_\eta = \mu_0 L^2 / \eta. \] (10.5)

This is called the resistive diffusion time.

We will soon see that the characteristic time scale associated with ideal (\( \eta = 0 \)) MHD processes is the Alfvén time
\[ \tau_A = \frac{L}{V_A}, \] (10.6)

where \( V_A^2 = B^2/\mu_0 \rho \) is the square of the Alfvén velocity. The ratio of the resistive and ideal MHD time scales is called the Lundquist number
\[ S = \frac{\tau_R}{\tau_A} = \mu_0 \frac{L V_A}{\eta}. \] (10.7)

It turns out that for many (but not all) MHD situations, \( S >> 1 \). The Lundquist number plays an important role in describing the dynamics of hot magnetized plasmas. We will return to the Lundquist number and its importance when we discuss magnetic reconnection later in this course.

We now inquire as to the overall effect of electrical resistivity on plasma confinement. We will discuss confinement in more detail when we discuss MHD equilibrium states. For now, we assume that we can attain a state of quasi-force balance in which the plasma is contained (or confined) within a magnetic field, with more plasma on the “inside” and more field on the “outside,” as sketched in Fig. 10.1

By quasi-force balance, we mean that the time on which the system evolves is more than the time required for wave propagation across the system (all concepts to be defined later), so that inertia \( (\rho d\mathbf{V}/dt) \) can be neglected in the equation of motion. This approach is difficult (but possible) to justify theoretically, but it is useful and we will adopt it here without justification.