We now return to ideal MHD, so that $\mathbf{E} + \mathbf{V} \times \mathbf{B} = 0$. The magnetic flux through and closed circuit $C$ is

$$\Phi = \int_S \mathbf{B} \cdot \hat{n} dS, \quad (12.1)$$

where $S$ is any surface bounded by $C$. Since $\nabla \cdot \mathbf{B} = 0$, we can write $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential. Then the flux can also be written as

$$\Phi = \int_S \nabla \times \mathbf{A} \cdot \hat{n} dS = \oint_C \mathbf{A} \cdot d\mathbf{l}. \quad (12.2)$$

Now consider the volume defined by all field lines enclosed by the curve $C$. This volume $V$ defines a flux tube. The flux $\Phi$ within $V$ is constant because $\mathbf{B}$ is everywhere tangent to its boundary. We know that, since $\nabla \cdot \mathbf{B} = 0$, the tube thus defined either closes on itself or fills space ergodically. Any finite volume $V_0$ contains an infinite number of such flux tubes.

Now consider the following integral:

$$K_l = \int_{V_l} \mathbf{A} \cdot \mathbf{B} dV, \quad (12.3)$$

where $V_l$ is the volume of the $l$th in $V$. The flux tube will move about with the fluid velocity $\mathbf{V}$. As it does, Eq. (12.3) changes according to

$$\frac{dK_l}{dt} = \int_{V_l} \left( \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} dV + \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{A} \cdot \mathbf{B} \frac{d}{dt} dV \right). \quad (12.4)$$
The last term is evaluated as
\[
\frac{d}{dt} V = \frac{d}{dt} dx_1 dx_2 dx_3 = V_1 dx_2 dx_3 + V_2 dx_2 dx_3 + V_3 dx_1 dx_2
\]
\[
= V \cdot \hat{n} dS. \tag{12.5}
\]

Then using Faraday’s law, we have
\[
\frac{dK_L}{dt} = \int_{V_l} \left( -E + \nabla \phi \right) \cdot B dV + \int_{V_l} A \cdot \left( -\nabla \times E \right) \cdot B dV + \int_{S_l} (A \cdot B) (V \cdot \hat{n}) dS, \tag{12.6}
\]
where \( \phi \) is the scalar potential. Now
\[
\nabla \cdot (A \times E) = E \cdot \nabla \times A - A \cdot \nabla \times E, \tag{12.7}
\]
so that the second integral can be written as
\[
\int_{V_l} A \cdot (\nabla \times E) \cdot B dV = \int_{V_l} E \cdot B dV - \int_{V_l} \nabla \cdot (A \times E) dV
\]
\[
= \int_{V_l} E \cdot B dV - \int_{S_l} (A \times E) \cdot \hat{n} dS. \tag{12.8}
\]
Similarly, the first integral can be rewritten as
\[
\int_{V_l} \nabla \phi \cdot B dV = \int_{V_l} \nabla \cdot (\phi B) dV
\]
\[
= \int_{S_l} \phi B \cdot \hat{n} dS = 0, \tag{12.9}
\]
because \( \nabla \cdot B = 0 \) and \( B \cdot \hat{n} = 0 \) on \( S_l \) by definition since \( V_l \) is a flux tube. Therefore
\[
\frac{dK_L}{dt} = -2 \int_{V_l} E \cdot B dV + \int_{S_l} (A \times E) \cdot \hat{n} dS + \int_{S_l} (A \cdot B) (V \cdot \hat{n}) dS. \tag{12.10}
\]
Now invoking ideal MHD, \( E = -V \times B \), Eq. (12.10) becomes
\[
\frac{dK_L}{dt} = - \int_{S_l} \left[ (A \cdot B) (V \cdot \hat{n}) - (A \cdot V) (B \cdot \hat{n}) \right] dS = 0, \tag{12.11}
\]