Lecture 2
Review of Scalars, Vectors, Tensors, and Dyads

Our vice always lies in the direction of our virtues, and in their best estate are but plausible imitations of the latter.
Henry David Thoreau

In MHD, we will deal with relationships between quantities such as the magnetic field and the velocity that have both magnitude and direction. These quantities are examples of vectors (or, as we shall soon see, pseudovectors). The basic concepts of scalar and vector quantities are introduced early in any scientific education. However, to formulate the laws of MHD precisely, it will be necessary to generalize these ideas and to introduce the less familiar concepts of matrices, tensors, and dyads. The ability to understand and manipulate these abstract mathematical concepts is essential to learning MHD. Therefore, for the sake of both reference and completeness, this lecture is about the mathematical properties of scalars, vectors, matrices, tensors, and dyads. If you are already an expert, or think you are, please skip class and go on to Lecture 3. You can always refer back here if needed!

A scalar is a quantity that has magnitude. It can be written as

\[ S \propto 9 \] (2.1)

It seems self-evident that such a quantity is independent of the coordinate system in which it is measured. However, we will see later in this lecture that this is somewhat naïve, and we will have to be more careful with definitions. For now, we say that the magnitude of a scalar is independent of coordinate transformations that involve translations or rotations.

A vector is a quantity that has both magnitude and direction. It is often printed with an arrow over it (as in \( \vec{V} \)) or in bold-face type (as in \( \mathbf{V} \), which is my preference). When handwritten, I use an underscore (as in \( \underline{V} \), although many prefer the arrow notation here, too). It can be geometrically represented as an arrow. A vector has a tail and a head (where the arrowhead is). Its magnitude is represented by its length. We emphasize that the vector has an “absolute” orientation in space, i.e., it exists independent of any particular coordinate system. Vectors are therefore “coordinate-free” objects, and expressions involving vectors are true in any coordinate system. Conversely, if an expression involving vectors is true in one coordinate system, it is true in all coordinate systems. (As with the scalar, we will be more careful with our statements in this regard later in this lecture.)
Vectors are added with the parallelogram rule. This is shown geometrically in Fig. 2.1.

**Fig. 2.1** Illustration of the parallelogram rule for adding vectors

This is represented algebraically as \( C = A + B \).

We define the *scalar product* of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) as

\[
\mathbf{A} \cdot \mathbf{B} = AB \cos \theta,
\]

(2.2)

where \( A \) and \( B \) are the magnitudes of \( \mathbf{A} \) and \( \mathbf{B} \) and \( \theta \) is the angle (in radians) between them, as in Fig. 2.2:

**Fig. 2.2** Illustration of the scalar product of two vectors \( \mathbf{A} \) and \( \mathbf{B} \) as the projection of one on the other

The quantity \( S = \mathbf{A} \cdot \mathbf{B} \) is the projection of \( \mathbf{A} \) on \( \mathbf{B} \), and vice versa. Note that it can be negative or zero. We will soon prove that \( S \) is a scalar.

It is sometimes useful to refer to a vector \( \mathbf{V} \) with respect to some coordinate system \( (x_1, x_2, x_3) \), as shown in Fig. 2.3. Here the coordinate system is orthogonal. The vectors \( \hat{e}_1, \hat{e}_2, \) and \( \hat{e}_3 \) have unit length and point in the directions of \( x_1, x_2, \) and \( x_3 \), respectively. They are called *unit basis vectors*. The *components* of \( \mathbf{V} \) with respect to \( (x_1, x_2, x_3) \) are then defined as the scalar products

\[
V_1 = \mathbf{V} \cdot \hat{e}_1, \quad V_2 = \mathbf{V} \cdot \hat{e}_2, \quad V_3 = \mathbf{V} \cdot \hat{e}_3.
\]

(2.3a,b,c)

The three numbers \( (V_1, V_2, V_3) \) also define the vector \( \mathbf{V} \).