Effects of 1-Greedy $S$-Metric-Selection on Innumerably Large Pareto Fronts

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Abstract. Evolutionary multi-objective algorithms (EMOA) using performance indicators for the selection of individuals have turned out to be a successful technique for multi-objective problems. Especially, the selection based on the $S$-metric, as implemented in the SMS-EMOA, seems to be effective. A special feature of this EMOA is the greedy ($\mu + 1$)-selection. Based on a pathological example for a population of size two and a discrete Pareto front it has been proven that a ($\mu + 1$)- (or 1-greedy) EMOA may fail in finding a population maximizing the $S$-metric. This work investigates the performance of ($\mu + 1$)-EMOA with small fixed-size populations on Pareto fronts of innumerable size. We prove that an optimal distribution of points can always be achieved on linear Pareto fronts. Empirical studies support the conjecture that this also holds for convex and concave Pareto fronts, but not for continuous shapes in general. Furthermore, the pathological example is generalized to a continuous objective space and it is demonstrated that also ($\mu + k$)-EMOA are not able to robustly detect the globally optimal distribution.

1 Introduction

The main question addressed in this work is concerned with the general suitability of a 1-greedy evolutionary multi-objective algorithm (EMOA) for the approximation of continuous Pareto fronts, which consist of an innumerable number of Pareto optimal solutions. As a 1-greedy EMOA, we denote a steady-state ($\mu + 1$)-EMOA that replaces only one individual by greedily selecting the $\mu$ best ones according to a preference relation (in the style of the definitions of $k$-greediness by Zitzler et al. [ZTB08]). The question is of special interest, since—for some time now—we advocate the use of an EMOA that adheres to the 1-greedy scheme using the $S$-metric or dominated hypervolume as preference relation, namely the SMS-EMOA [BNE07]. In contrast to other EMOA (e.g. NSGA-II [DPAM02]), it accepts only one new individual per generation in order to monotonically improve the quality of the Pareto front approximation. Naturally, one can ask if exchanging only one individual at a time is sufficient to avoid getting stuck in non-optimal configurations. However, past experience with the SMS-EMOA has
nourished the belief that this algorithm is capable of coping with all practically relevant situations, although a general proof in either direction is missing. This kind of general proof, even if restricted to continuous Pareto fronts, is sophisticated, unless impossible. Thus, the aim of this paper is to gradually phase this task using both case-related formal proofs and empirical studies.

Recently, a simple discrete counter-example has been provided, which proved that the 1-greedy scheme based on the dominated hypervolume can fail [ZTB08] (cf. Sec. 2.3). However, the example is extreme in many aspects: It employs a population of only two individuals on a Pareto front of four points. Thus, we would like to know to what extend this phenomenon occurs in more realistic scenarios. We are interested in continuous Pareto fronts and show that the discrete counter example can easily be extended into the (piecewise) continuous domain, with the essential property still holding: For most initializations, a 1-greedy EMOA will fail to obtain the optimal distribution of points on that Pareto front. To further investigate the cause of failure, we optimize the S-metric value of the population directly using a (1, 5)- and a (5, 10)-CMA-ES (Covariance Matrix Adaptation Evolution Strategy [HO01]). Our results show that not only 1-greedy EMOA, but also non-elitist EMOA with $\lambda > \mu$ can fail with high probability. This indicates that the problem is indeed very hard since the local optimum is a strong attractor for any kind of optimizer. Studying the structure of the problem, we give generalizing conjectures on the interrelationship of the Pareto front and greediness.

Given the successful applications and assuming that the failure on the mentioned counter examples stem from the extreme constitution of the Pareto front, we investigate the properties on connected simpler shapes. For linear Pareto fronts, it is proved that a 1-greedy hypervolume selection scheme is sufficient to reach the optimal distribution of points with respect to the dominated hypervolume. Regarding convex Pareto fronts, we show that the problem of maximizing the hypervolume with a given number of points is not concave, otherwise the 1-greediness would hold directly (cf. Sec. 4.1). However, the concavity is not a necessary condition for 1-greediness. We perform empirical studies on Pareto fronts of different curvature, which demonstrate that even a simplified SMS-EMOA reaches the global optimum showing that the problem is solvable by a 1-greedy EMOA. Furthermore, these studies give counter-intuitive insights on the optimal distributions of the points and their corresponding hypervolume contributions, i.e., the amount that is disjointly dominated by a point and is lost when the point is removed [BNE07].

The paper is structured as follows. In section 2 the basic definitions, which are used in this paper, are provided and the discrete pathological example for 1-greedy indicator-based EMOA is recapitulated. The continuous variant of the example is derived and, together with another problem with disconnected fronts, empirically studied in section 3. Afterwards, we focus on continuous Pareto fronts by analyzing and empirically studying connected fronts of different curvature in section 4. For simple cases, also formal proofs are provided. Finally, the paper is summarized and the important results are concluded in section 5.