Building a Learning space

An application of either QUERY or its extension PS-QUERY results in a knowledge space, that is, a structure closed under union which is not necessarily a learning space. In many practical situations, however, the essential properties of learning spaces are regarded as crucial. In particular, the fringes enables a compact, precise delineation of any knowledge state (cf. Theorems 4.1.7 and 2.2.4(iii)). This property plays a key role in providing a meaningful summary of an assessment. Moreover, in the guise of the outer fringe, it opens the path to further learning. This raises the following problem: assuming that, except for errors, the responses to the queries are dictated by a latent learning space \( L \), can a learning space approximating \( L \) be derived by the querying method through some elaboration of \( \text{QUERY} \)? In this chapter, we outline two quite different procedures to achieve this goal.

The first one is a modification of \( \text{QUERY} \) inspired by the fact, established by Theorem 2.2.4, that a knowledge space \((Q, \mathcal{K})\) is a learning space if and only if it satisfies Axiom [MA] for an antimatroid:

[MA] If \( K \) is a nonempty subset of \( Q \) belonging to the family \( \mathcal{K} \), then there is some \( q \) in \( K \) such that \( K \setminus \{q\} \) is a state of \( \mathcal{K} \).

This axiom suggests the following revision of \( \text{QUERY} \). We start with a collection of (potential) states which forms some initial learning space \( L_0 \). For example, \( L_0 \) could be the power set of \( Q \), or an ordinal space obtained from implementing Block 1 of \( \text{QUERY} \). (We know by Theorem 4.1.10 that ordinal spaces are learning spaces.) We assume that the responses to the queries are in principle based on a latent learning space \( L \subseteq L_0 \). Whenever a positive response to a query \((A, q)\) is observed, we delete from the current learning space, starting with \( L_0 \), all the states contradicting this response only when the resulting structure satisfies Axiom [MA]. A simple test to this effect will be given in Theorem 16.1.6.

The defect of the above test is that it involves the whole collection of states which can be unmanageably large. Hence, the test and the whole procedure may not be applicable in many practical cases\(^1\).

\(^1\) The original \( \text{QUERY} \) procedure avoids this drawback by storing an entailment rather than the whole collection of (potential) states.
Nevertheless, the idea is a sound one and it can be adapted: rather than removing states from the learning space itself, we can operate in a similar fashion on the base of the learning space or on the surmise function, both of which are typically much smaller structures. The relevant result is Condition (iii) in Theorem 5.4.1, which is akin to Axiom [MA]. This condition is recalled below in the form of an axiom for learning spaces.

\[ \text{[L3]} \quad \text{For any clause } C \text{ for an item } \{ r \} \text{ in a knowledge space } \mathcal{K}, \text{ the set } C \setminus \{ r \} \text{ is a state of } \mathcal{K}. \]

Rephrasing a previous result in terms of [L3], we have then:

**Theorem 5.4.1(iii).** A knowledge space is a learning space if and only if it satisfies Axiom [L3].

This observation leads to an algorithm managing the surmise function appropriately. At the last stage of the algorithm, a learning space is constructed as the collection spanned by the clauses in the final surmise function. We describe this algorithm in Section 16.2.

The second procedure for building a learning space by the querying procedure, which is due to David Eppstein (see Eppstein, Falmagne, and Uzun, 2009; Eppstein, 2010), is quite different from the first one, which essentially (if not literally) proceed by a gradual elimination of potential states. In a first step, a knowledge space is built by a standard application of \texttt{QUERY} or \texttt{PS-QUERY}. A learning space is then constructed by judiciously adding states until wellgradedness is satisfied, while preserving \( \cup \)-closure. We sketch Eppstein method in Section 16.3.

### 16.1 Preparatory Concepts and an Example

Axiom [MA] suggests the concept of a ‘critical’ state, whose removal would result in a violation of the axiom. This concept is defined below, together with two related ones.

#### 16.1.1 Definition. A nonempty state \( L \) in a knowledge structure \( \mathcal{K} \) is hanging if its inner fringe \( L^I \) is empty\(^2\). The state \( L \) is almost hanging if it contains more than one item, but its inner fringe consists of a single item. Denoting the latter item by \( p \), we then say that the state \( L \setminus \{ p \} \) is critical in \( \mathcal{K} \) for \( L \).

So, an almost hanging state defines exactly one critical state. However, a state may be critical for several almost hanging states.

#### 16.1.2 Example. Consider the knowledge space with domain \( Q = \{ a, b, c, d \} \) and collection of states

\[
\mathcal{L} = \{ \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, c \}, \{ a, d \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \}, Q \}.
\]

\(^2\) We recall that the inner fringe \( L^I \) of a state \( L \) is the set of all items \( q \) in \( L \) such that \( L \setminus \{ q \} \) is a state; cf. Definition 4.1.6.