Chapter 6
Augmented Time-Stepping by Step Size Adjustment and Extrapolation

This chapter investigates how the accuracy of Moreau’s midpoint rule can be increased by step size adjustment and extrapolation. The idea is to perform time steps which contain switching points with a minimal step size $\Delta t_{\text{min}}$. Smooth time steps, i.e. time steps which do not contain switching points, are processed with larger step sizes. Furthermore, extrapolation methods are used to increase the integration order in these time steps. The integration order of the classical Moreau midpoint rule is analyzed in Sect. 6.1. In Sect. 6.2 it is discussed how switching points can be localized, which yields a step size controlled time-stepping algorithm. Finally, in Sect. 6.3 extrapolation methods are applied to Moreau’s midpoint rule, which requests some restrictions on these methods. Some examples are given in Sect. 6.5. The chapter is based on the original publications [92, 94].

6.1 Integration Order of Moreau’s Midpoint Rule

This section aims at analyzing the integration order of Moreau’s time-stepping scheme. Similar to the classical approach, the integration order is obtained by comparing the expansions of the exact and numeric solution with respect to the time $t$. The expansion of a non-smooth time step is obtained by combining the expansions of the associated smooth parts within the time step. Note that the sketched approach to obtain the integration order must be seen from an engineering point of view, i.e. it can not compete with much more sophisticated mathematical convergence proofs [28, 63].

6.1.1 Definition of the Integration Order

Consider a system of ordinary differential equations $\dot{y} = f(y, t)$ with initial conditions $y(t_0) = y_0$, of which an exact solution is $y(t)$. Numerical schemes of the form
\[ y_{s+1} = y_s + \Delta t \varphi(t_{s+1}, t_s, y_s) \]  

(6.1)

provide a sequence of values \( y_s \) which approximate the exact solution \( y(t_s) \) \cite{[86]}. The function \( \varphi(t_{s+1}, t_s, y_s) \) is called incremental function of the integration scheme. The local integration error is defined as

\[ e_L(t_{s+1}) = y(t_{s+1}) - \hat{y}_{s+1} = y(t_{s+1}) - y(t_s) - \Delta t \varphi(t_{s+1}, t_s, y_s), \]

(6.2)

in which \( \hat{y}_{s+1} \) is the approximation obtained from one integration step starting from the exact value \( y(t_s) \). The terms \( y(t_{s+1}) \) and \( \hat{y}_{s+1} \), which build the local integration error, can be written as a function of \( y(t_s) \) by using a Taylor-series expansion, provided that the trajectories are smooth. Comparing these expansions yields an estimation for the local integration error,

\[
\max_{t_i} ||e_L(t_i)|| \leq K_L \Delta t^{p+1} = O(\Delta t^{p+1}),
\]

(6.3)

in which \( K_L \) is an arbitrary bounded constant, which is independent of \( \Delta t \). The scalar \( p \) is called the integration order. The global integration error gathers the accumulated local errors during the integration,

\[ e_G(t_{s+1}) = y(t_{s+1}) - y_{s+1} = y(t_{s+1}) - y_s - \Delta t \varphi(t_{s+1}, t_s, y_s), \]

(6.4)

in which \( y_{s+1} \) is the approximation obtained after several integration steps. In the following, the global integration error \( e_G \) is expressed as sum of the local integration errors \( e_L \). It holds that

\[
||e_G(t_{s+1})|| = ||y(t_{s+1}) - y_{s+1}|| = ||y(t_{s+1}) - \hat{y}_{s+1} + \hat{y}_{s+1} - y_{s+1}||
\]

\[
= ||y(t_{s+1}) - \hat{y}_{s+1} + y(t_s) - y_s + \Delta t (\varphi(t_{s+1}, t_s, y(t_s)) - \varphi(t_{s+1}, t_s, y_s))||
\]

\[
\leq ||e_L(t_{s+1})|| + (1 + L_\varphi \Delta t) ||e_G(t_s)||
\]

\[
\leq ||e_L(t_{s+1})|| + e^{L_\varphi \Delta t} ||e_G(t_s)||
\]

(6.5)

where \( L_\varphi \) is the Lipschitz constant of \( \varphi \),

\[
||\varphi(t_{s+1}, t_s, y(t_s)) - \varphi(t_{s+1}, t_s, y_s)|| \geq L_\varphi ||y(t_s) - y_s|| = L_\varphi ||e_G(t_s)||.
\]

(6.6)

The relation (6.5) allows for estimating the global error \( e_G(t_{s+1}) \) by the global error \( e_G(t_s) \) and the local error \( e_L(t_{s+1}) \). As a consequence, the global error \( e_G(t_{s+1}) \) can also be estimated by a sum of local errors,

\[
||e_G(t_{s+1})|| \leq \sum_{i=1}^{s+1} \left( e^{L_\varphi (s+1-i) \Delta t} ||e_L(t_i)|| \right) \leq e^{L_\varphi (t_{s+1} - t_0)} \sum_{i=1}^{s+1} ||e_L(t_i)||,
\]

(6.7)