Chapter 2
The Evolution Formalism

2.1 Space plus time decomposition

The general covariant approach to general relativity is not adapted to our experience from everyday life. The most intuitive concept is not that of space-time geometry, but rather that of a time succession of space geometries. This ‘flowing geometries’ picture could be easily put into the computer, by discretizing the time coordinate, in the same way that the continuous time flow of the real life is coded in terms of a discrete set of photograms in a movie.

In this sense, we can say that general relativity theory, when compared with other physical theories like electromagnetism, has been built upside down. In Maxwell theory one starts with the everyday concepts of electric charges, currents, electric and magnetic fields. One can then write down a (quite involved) set of field equations, Maxwell equations, that can be easily interpreted by any observer. Only later some ‘hidden symmetry’ (Lorentz invariance) of the solution space is recognized, and this allows to rewrite Maxwell equations in a Lorentz-covariant form. But the price to pay is gluing charges and currents on one side, and electric and magnetic fields on the other, into new 4D objects that obscure the direct relation to experience the original (3D) pieces.

In general relativity, we have started from the top, at the 4D level, so we must go downhill, in the opposite sense:

- by selecting a specific (but generic) time coordinate;
- by decomposing every 4D object (metric, Ricci, and stress–energy tensors) into more intuitive 3D pieces;
- by writing down the (much more complicated) field equations that translate the manifestly covariant ones (1.26) in terms of these 3D pieces.

General covariance will then become a hidden feature of the resulting ‘3+1 equations.’ The equations themselves will no longer be covariant under a general coordinate transformation. But, as the solution space is not being modified by the decomposition, general coordinate transformations will still
map solutions into solutions (as it happens with Lorentz transformations in Maxwell equations). The underlying invariance of the equations under general coordinate transformations is then preserved when performing the 3+1 decomposition. General covariant 4D equations just show up this invariance in an explicit way.

2.1.1 A prelude: Maxwell equations

Maxwell equations are usually written as

\[
\begin{align*}
\nabla \cdot E &= 4\pi q, \\
\nabla \cdot B &= 0, \\
-\partial_t E + \nabla \times B &= 4\pi J, \\
\partial_t B + \nabla \times E &= 0,
\end{align*}
\]

where it is clear that the charge and current densities, \(q\) and \(J\), act as the sources of the electric and magnetic fields, \(E\) and \(B\). We assume here for simplicity the vanishing of both the electric and magnetic susceptibilities, so that \(D = E\) and \(B = H\).

The second pair of equations (2.3) and (2.4) can be interpreted as providing a complete set of time evolution equations for the electric and magnetic fields (evolution system), whereas the first two equations (2.1) and (2.2) do not contain time derivatives and can be interpreted then as constraints. A straightforward calculation shows that these constraints are first integrals of the evolution system, as a consequence of the charge continuity equation:

\[
\partial_t q + \nabla \cdot J = 0.
\]

Now, we can start joining pieces. The charge and current densities can be combined to form a four-vector \(I^\mu\),

\[
I^\mu \equiv (q, J).
\]

The electric and magnetic fields can be combined in turn to form an antisymmetric tensor, namely

\[
F_{\mu\nu} \equiv \begin{pmatrix} 0 & -E_j \\ E_i & F_{ij} \end{pmatrix}, \quad F_{ij} \equiv \epsilon_{ijk}B^k
\]

(electromagnetic field tensor).

The pair (2.1) and (2.3) of Maxwell equations can then be written in the manifestly covariant form

\[
\nabla_\nu F^{\mu\nu} = 4\pi I^\mu,
\]