In this chapter we describe selected numerical integration methods that can be applied to general MVN and MVT problems. Such problems are often given as a multiple integral over an unbounded integration domain. The application of most integration methods requires a transformation or reparameterization of the original problem to one where the integration domain is bounded. Consequently, we first discuss various reparameterizations in Section 4.1. In Section 4.2 we then describe several multidimensional integration methods.

4.1 Reparameterizations

4.1.1 Spherical-Radial Transformation Methods

A key issue for many numerical integration methods is a suitable reparameterization of the original integral. The aim is to transform mode and scale appropriately, thus making the integrand more suitable for integration. In addition, efficient integration methods may require the variables to range over a particular set. The first class of methods that we discuss uses the combination of an initial standardization with a spherical-radial (SR) transformation. These reparameterizations have been developed independently by several authors for MVN integrals (Deák, 1980, 1990; Richter, 1994; Monahan and Genz, 1997; Somerville, 1997, 1998; Ni and Kedem, 1999). Similar derivations for MVT problems starting from either of equations (1.2) to (1.4) are given by Somerville (1997, 1998) and Genz and Bretz (2002).

All of the methods begin with a Cholesky decomposition. This yields a lower triangular \( k \times k \) matrix \( L = (l_{ij}) \) such that \( l_{ii} > 0 \) and \( L^{-1} \Sigma L^{-t} = I_k \). Let \( |A| \) denote the determinant of a matrix \( A \). Then the linear transformation \( x = Ly \), with its Jacobian given by

\[
dx = |L|dy = \sqrt{|\Sigma|}dy,
\]

reduces equation (1.1) to
$\Phi_k(a \leq x \leq b; \Sigma) = \Phi_k(a \leq Ly \leq b; \mathbf{I}_k)$

\[(4.1)\]

where $\mathbf{I}_j$ is the $j$-th row of $\mathbf{L}$. Note that other methods exist (for example, the principal component decomposition) which lead to similar standardizations.

A second change of variables transforms the standardized vector $\mathbf{y}$ to a radius $r$ and direction vector $\mathbf{z}$, $\mathbf{y} = rz$ with $||z|| = 1$, so that $\mathbf{y}^t \mathbf{y} = r^2$ for $r \geq 0$. This changes the integral (4.1) accordingly to

$\Phi_k(a, b; \Sigma) = \int_{||z|| = 1} \frac{2^{1-\frac{k}{2}}}{\Gamma(k)} \int_{r_l(z)}^{r_u(z)} r^{k-1} e^{-\frac{r^2}{\nu}} dr dU(z), \quad (4.2)$

where $U(.)$ is the joint cdf of the uniform distribution on the unit hypersphere $\{z : ||z|| = 1\}$. Letting $R(z) = \{r : r \geq 0, a \leq rLz \leq b\}$, the integration limits of the inner integral are defined as $r_l(z) = \min \{r : r \in R(z)\}$ and $r_u(z) = \max \{r : r \in R(z)\}$. If we let $\mathbf{v} = \mathbf{Lz}$, the limits for the $r$-variable integration can be given more explicitly by

$r_l(z) = \max \left\{0, \max_{v_i > 0} \{a_i/v_i\}, \max_{v_i < 0} \{b_i/v_i\}\right\},$

$r_u(z) = \max \left\{0, \min_{v_i > 0} \{b_i/v_i\}, \min_{v_i < 0} \{a_i/v_i\}\right\}.$

These limits are the distances from the origin to the two points where the vector with direction $\mathbf{z}$ intersects the boundary of the integration region.

For the MVT case, the expressions remain similar, with the exception of integrating a density for the $F_{k, \nu}$ distribution along the direction $\mathbf{z}$, instead of the $\chi_k$ distribution. We have

$T_k(a, b; \Sigma, \nu) = \int_{||z|| = 1} F_{k, \nu}(r_U(z)) - F_{k, \nu}(r_l(z)) dU(z), \quad (4.3)$

with

$F_{k, \nu}(r) = \frac{2\Gamma(\frac{\nu+k}{2})}{\Gamma(k)\Gamma(\frac{\nu}{2})\nu^{\frac{k}{2}}} \int_0^r \frac{t^{k-1}}{(1 + \frac{t^2}{\nu})^{\frac{k+\nu}{2}}} dt.$

This can also be written in terms of a $\text{Beta}_{a,b}$ distribution using the relation $F_{k, \nu}(r) = \text{Beta}_{\frac{k}{2}, \frac{\nu}{2}}(r^2/(\nu + r^2)).$

Extensions of these transformations were provided by Somerville (2001) for elliptical regions and by Lohr (1993) for general star shaped regions. Different