Chapter 3
Stochastic Volatility Models

In this chapter we discuss stochastic volatility models that play a crucial role in smile modeling not only on the theoretical side, but also on the practical side. Stochastic volatility models are extensively examined by financial researchers, and widely used in investment banks and financial institutions. In many trading floors, this type of option pricing model has replaced the Black-Scholes model to be the standard pricing engine, especially for exotic derivatives. Intuitively, it is a natural way to capture the volatility smile by assuming that volatility follows a stochastic process. Simple observations propose that the stochastic process for volatility should be stationary with some possible features such as mean-reverting, correlation with stock dynamics. In this sense, a mean-reverting square root process and a mean-reverting Ornstein-Uhlenbeck process adopted from the interest rate modeling are two ideal candidate processes for stochastic volatilities. Heston (1993) specified stochastic variances with a mean-reverting square root process and derived a pioneering pricing formula for options by using CFs. Stochastic volatility model with a mean-reverting Ornstein-Uhlenbeck process is examined by many researchers. Schöbel and Zhu (1999) extended Stein and Stein’s (1991) solution for a zero-correlation case to a general non-zero correlation case. In this chapter, we will expound the basic skills to search for a closed-form formula for options in a stochastic volatility model. Generally, we have two approaches available: the PDE approach and the expectation approach, both are linked via the Feynman-Kac theorem. However, as demonstrated in following, the expectation approach utilizes the techniques of stochastic calculus and is then an easier and more effective way. To provide an alternative stochastic volatility model, we also consider a model with stochastic variances specified as a mean-reverting double square root process.

3.1 Introduction

Since the Black-Scholes formula was derived, a number of empirical studies have concluded that the assumption of constant volatility is inadequate to describe stock
returns, based on two findings: (1) volatilities of stock returns vary over time, but persist in a certain level (mean-reversion property). This finding can be traced back to the empirical works of Mandelbrot (1963), Fama (1965) and Blattberg and Gonedes (1974) with the results that the distributions of stock returns are more leptokurtic than normal; (2) volatilities are correlated with stock returns, and more precisely, they are usually inversely correlated. Furthermore, volatility smile provides a direct evident for inconsistent volatility pattern with moneyness in the Black-Scholes model. Black (1976a), Schmalensee and Trippi (1978), Beckers (1980) investigated the time-series property between stock returns and volatilities, and found an imperfect negative correlation. Bakshi, Cao and Chen (BCC, 1997) and Nandi (1998) also reported a negative correlation between the implied volatilities and stock returns. Moreover, Beckers (1983), Pozerba and Summers (1984) gave evidence that shocks to volatility persist and have a great impact on option prices, but tend to decay over time. These uncovered properties associated with volatility such as leptokurtic distribution, correlation, mean-reversion and persistence of shocks, should be considered in a suitable option pricing model.

In order to model the variability of volatility and to capture the volatility smile, several approaches have been suggested. One approach was the so-called constant elasticity of variance diffusion model developed by Cox (1975), and may be regarded as a representative of a more general model class labeled to local volatility model. Cox assumed that volatility is a function of the stock price with the following form:

\[ v(S(t)) = aS(t)^{\delta - 1}, \quad \text{with} \quad a > 0, \quad 0 \leq \delta \leq 1. \]  

(3.1)

Since \( v(S(t)) \) is a decreasing function of \( S(t) \), volatilities are inversely correlated with stock returns. However, this deterministic function can not describe other desired features of volatility. Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) hypothesized that volatility is a deterministic function of the stock price and time, and developed a deterministic volatility function (DVF), with which they attempt to fit the observed cross-section of option prices exactly. This approach, as reported by Dumas, Fleming and Whaley (1998), does not perform better than an ad hoc procedure that merely smooths implied volatilities across strike prices and times to maturity.

A more general approach is to model volatility by a diffusion process and has been examined Johnson and Shanno (1987), Wiggins (1987), Scott (1987), Hull and White (1987), Stein and Stein (1991), Heston (1993), Schöbel and Zhu (1999) and Lewis (2000). The models following this approach are the so-called stochastic volatility models. Table (3.1) gives an overview of some representatives of stochastic volatility models.

There are no known closed-form option pricing formulas in the case of non-zero correlation between volatilities and stock returns for all models listed in Table 3.1 except for model (5) [Schöbel and Zhu, 1999], model (6) [Heston, 1993] as well as model (7). Models (1) and (3) perform no mean-reversion property and therefore can not capture the effects of shocks to volatility in the valuation of options. Models (1), (2), (3), (4) and (8) are not stationary processes and violate the feature of stationarity