RESTRUCTURING LATTICE THEORY: AN APPROACH BASED ON HIERARCHIES OF CONCEPTS

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ABSTRACT

Lattice theory today reflects the general status of current mathematics: there is a rich production of theoretical concepts, results, and developments, many of which are reached by elaborate mental gymnastics; on the other hand, the connections of the theory to its surroundings are getting weaker and weaker, with the result that the theory and even many of its parts become more isolated. Restructuring lattice theory is an attempt to reinvigorate connections with our general culture by interpreting the theory as concretely as possible, and in this way to promote better communication between lattice theorists and potential users of lattice theory.

The approach reported here goes back to the origin of the lattice concept in nineteenth-century attempts to formalize logic, where a fundamental step was the reduction of a concept to its "extent". We propose to make the reduction less abstract by retaining in some measure the "intent" of a concept. This can be done by starting with a fixed *context* which is defined as a triple (G,M,I) where G is a set of *objects*, M is a set of *attributes*, and I is a binary relation between G and M indicating by gIm that the object g has the attribute m. There is a natural Galois connection between G and M defined by $A' = \{m \in M \mid gIm$ for all $g \in A\}$ for $A \subseteq G$ and $B' = \{g \in G \mid gIm$ for all $m \in B\}$ for $B \subseteq M$. Now, a *concept* of the context (G,M,I) is introduced as a pair (A,B) with $A \subseteq G$, $B \subseteq M$, $A' = B$, and $B' = A$, where A is called the *extent* and B the *intent* of the concept (A,B). The *hierarchy of concepts* given by the relation subconcept-superconcept is captured by the definition $(A_1,B_1) \leq (A_2,B_2) \Leftrightarrow A_1 \subseteq A_2 (\Leftrightarrow B_1 \supseteq B_2)$ for concepts (A_1,B_1) and (A_2,B_2) of (G,M,I). Let $L(G,M,I)$ be the

I. Rival (ed.), Ordered Sets, 314-339.

set of all concepts of (G,M,I). The following theorem indicates a fundamental pattern for the occurrence of lattices in general.

THEOREM: *Let* (G,M,I) *be a context. Then* $(\underline{L}(G,M,I),\leq)$ *is a complete lattice* (called the *concept lattice* of (G,M,I)) *in which infima and suprema can be described as follows:*

$$\bigwedge_{i\in J}(A_i,B_i) = \left(\bigcap_{i\in J} A_i, \left(\bigcap_{i\in J} A_i\right)'\right),$$

$$\bigvee_{i\in J}(A_i,B_i) = \left(\left(\bigcap_{i\in J} B_i\right)', \bigcap_{i\in J} B_i\right).$$

Conversely, if L *is a complete lattice then* $L \cong (\underline{L}(G,M,I),\leq)$ *if and only if there are mappings* $\gamma : G \to L$ *and* $\mu : M \to L$ *such that* γG *is supremum-dense in* L, μM *is infimum-dense in* L, *and* gIm *is equivalent to* $\gamma g \leq \mu m$ *for all* $g \in G$ *and* $m \in M$; *in particular,* $L \cong (\underline{L}(L,L,\leq),\leq)$.

Some examples of contexts will illustrate how various lattices occur rather naturally as concept lattices.

(i) $(S,S,\neq)$ where S is a set.
(ii) $(\mathbb{N},\mathbb{N},1)$ where $\mathbb{N}$ is the set of all natural numbers.
(iii) $(V,V^*,\perp)$ where V is a finite-dimensional vector space.
(iv) $(V,\mathrm{Eq}(V),\models)$ where V is a variety of algebras.
(v) $(G\times G,\mathbb{R}^G,\sim)$ where G is a set of objects, $\mathbb{R}^G$ is the set of all real-valued functions on G, and $(g_1,g_2)\sim\alpha$ iff $\alpha g_1 = \alpha g_2$.

Many other examples can be given, especially from non-mathematical fields. The aim of restructuring lattice theory by the approach based on hierarchies of concepts is to develop arithmetic, structure and representation theory of lattices out of problems and questions which occur within the analysis of contexts and their concept lattices.