Explicit Algebraic Solution of Geometrically Simple Serial Manipulators

Martin Pfurner

Abstract An algorithm is developed to solve the inverse kinematics of special serial manipulators that contain a spherical or planar sub-chain anywhere within an entire six joint sequence. It is known, for such cases, that the inverse kinematics is solvable in closed form, i.e., with a univariate polynomial of degree four or less; sometimes even with a quadratic equation. This algorithm yields explicit algebraic solutions for these kind of manipulators even when the design or the end-effector pose is not explicitly given.

1 Introduction

In the late 1980s it was shown that a general six revolute serial chain admits up to sixteen configurations to attain the same given end-effector pose, i.e. position and orientation of a frame attached to the end-effector of the mechanism (see [5], [8]). That means, in an analytical way, that one has to solve a univariate polynomial of degree sixteen to achieve all the solutions of the inverse kinematics problem. But there exist manipulators having special design which reduce this number of configurations as presented in [7].

[1] thoroughly treats the inverse kinematics of decoupled manipulators. These manipulators are defined in this book as manipulators having a wrist, i.e., whose last three axes intersect in a point. The algorithm to be introduced herein allows the wrist to be placed anywhere in the 6R chain.

Moreover similar reduction in the univariate polynomial degree can be achieved with an absoluter common point of axial intersection, i.e., a point at infinity so as to form a planar 3R-sub-chain.

In many publications the phrase solvable in closed form is used for such manipulators. In [6] such manipulators are denoted as analytical manipulators, what

Martin Pfurner
Unit Geometry and CAD, University of Innsbruck, Austria e-mail: Martin.Pfurner@uibk.ac.at
means that the inverse kinematics of this manipulators could be solved without using numerical solvers. This paper shows an algorithm that has the potential to solve some mechanisms in closed form. Here this means without specifying any Denavit Hartenberg parameters except the one for the wrist or the planar part.

The design of the mechanism will be given in Denavit-Hartenberg notation. To use the advantages of algebraic geometry (see [2] and [3]) and modern computer algebra systems in order to solve systems of polynomial equations all the angles we use are interchanged by the tangent half of their values. This makes algebraic equations out of equations including sine and cosine functions. In the following formulas the twist angles \( \alpha_i \) between adjacent axes are represented by their algebraic values \( a_l_i, i = 1, \ldots, 6 \), using the tangent half substitution \( a_l_i = \tan(\alpha_i/2) \).

2 Inverse Kinematics

For the inverse kinematics of the 6R-chain we will use a slightly modified version of the algorithm developed in [4]. The difference is the way of decomposing the 6R-chain into two sub-chains. The next sections will give a short outlook of the algorithm in its original version an in the modified one.

2.1 Original Algorithm

In the algorithm presented in [4] the manipulator is cutted, mathematically, between the third and fourth axis to achieve two open 3R-chains. One is fixed in the base of the original mechanism, the other one is fixed in the prescribed end-effector pose for this inverse kinematics. Before the decomposition two copies of a coordinate frame were attached to the resulting sub-chains. One, namely \( \Sigma_L \) is fixed to the first and the other one \( \Sigma_R \) to the second sub-chain in a way, that these frames coincide. They will then act as the end-effector frames of the resulting 3R-sub-chains. The task after cutting is to find all possible poses of \( \Sigma_L \) and \( \Sigma_R \) where they coincide. There the 6R-chain can be connected again. These configurations of the subchains yield all possible configurations where the end-effector of the whole mechanism is able to reach this prescribed pose. In the algorithm presented in [4] we use kinematic mapping to find solutions for this problem. It maps the end effector pose of both 3R-sub-chains onto manifolds in a seven dimensional image space \( \mathbb{P}^7 \), the kinematic image space. These manifolds are the so called constrained manifolds of the chains. In the case of a general 3R-chain it is the intersection of a Segre Manifold with the Study Quadric \( S^2_6 \) with

\[
S^2_6 : \quad x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 = 0,
\]  

(1)