Chapter 2
Continuous Time Normal Martingales

This chapter is concerned with the basics of stochastic calculus in continuous time. In continuation of Chapter 1 we keep considering the point of view of normal martingales and structure equations, which provides a unified treatment of stochastic integration and calculus that applies to both continuous and discontinuous processes. In particular we cover the construction of single and multiple stochastic integrals with respect to normal martingales and we discuss other classical topics such as quadratic variations and the Itô formula.

2.1 Normal Martingales

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space equipped with a right-continuous filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), i.e. an increasing family of sub \(\sigma\)-algebras of \(\mathcal{F}\) such that

\[ \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s, \quad t \in \mathbb{R}_+. \]

We refer to Section 9.5 in the Appendix for recalls on martingales in continuous time.

Definition 2.1.1. A square-integrable martingale \((M_t)_{t \in \mathbb{R}_+}\) such that

\[ \mathbb{E} \left[ (M_t - M_s)^2 | \mathcal{F}_s \right] = t - s, \quad 0 \leq s < t, \]  

(2.1.1)

is called a normal martingale.

Every square-integrable process \((M_t)_{t \in \mathbb{R}_+}\) with centered independent increments and generating the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) satisfies

\[ \mathbb{E} \left[ (M_t - M_s)^2 | \mathcal{F}_s \right] = \mathbb{E} \left[ (M_t - M_s)^2 \right], \quad 0 \leq s \leq t, \]

hence the following remark.

Remark 2.1.2. A square-integrable process \((M_t)_{t \in \mathbb{R}_+}\) with centered independent increments is a normal martingale if and only if
\[ E \left[ (M_t - M_s)^2 \right] = t - s, \quad 0 \leq s \leq t. \]

In our presentation of stochastic integration we will restrict ourselves to normal martingales. As will be seen in the next sections, this family contains Brownian motion and the standard Poisson process as particular cases.

Remark 2.1.3. A martingale \((M_t)_{t \in \mathbb{R}_+}\) is normal if and only if \((M_t^2 - t)_{t \in \mathbb{R}_+}\) is a martingale, i.e.
\[ E \left[ M_t^2 - t | F_s \right] = M_s^2 - s, \quad 0 \leq s < t. \]

Proof. This follows from the equalities
\[
\begin{align*}
E \left[ (M_t - M_s)^2 | F_s \right] &= (t - s) \\
&= E \left[ M_t^2 - M_s^2 - 2(M_t - M_s)M_s | F_s \right] - (t - s) \\
&= E \left[ M_t^2 - M_s^2 | F_s \right] - 2M_s E \left[ M_t - M_s | F_s \right] - (t - s) \\
&= E \left[ M_t^2 | F_s \right] - t - (E \left[ M_s^2 | F_s \right] - s).
\end{align*}
\]

Throughout the remainder of this chapter, \((M_t)_{t \in \mathbb{R}_+}\) will be a normal martingale and \((F_t)_{t \in \mathbb{R}_+}\) will be the right-continuous filtration generated by \((M_t)_{t \in \mathbb{R}_+}\), i.e.
\[ F_t = \sigma(M_s : 0 \leq s \leq t), \quad t \in \mathbb{R}_+. \]

2.2 Brownian Motion

In this section and the next one we present Brownian motion and the compensated Poisson process as the fundamental examples of normal martingales. Stochastic processes, as sequences of random variables can be naturally constructed in an infinite dimensional setting. Similarly to Remark 1.4.1 where an infinite product of discrete Bernoulli measures is mapped to the Lebesgue measure, one can map the uniform measure on “infinite dimensional spheres” to a Gaussian measure, cf. e.g. [85], [49], and references therein. More precisely, the surface of the \(n\)-dimensional sphere with radius \(r\) is
\[ s_n(r) = \frac{n \pi^{n/2}}{\Gamma(n/2 + 1)} r^{n-1} \sim \pi^{n/2} \sqrt{2\pi} e^{n/2} \left( \frac{n}{2} \right)^{-n/2} r^{n-1}, \]
where the equivalence is given by Stirling’s formula as \(n\) goes to infinity. The set of points on the sphere \(S_n(\sqrt{n})\) whose first coordinate \(x_1\) lies between \(a\)