Chapter 4
Microlocal Analysis

Alexander Strohmaier

Microlocal Analysis deals with the singular behavior of distributions in phase space.

4.1 Introduction

Distributions appear in physics in various forms: as mass distributions of point particles, as Green's functions, and as propagators in quantum field theory. The theory of distributions not only has applications such as these in physics but is also widely and intensively used in mathematics, in particular in the theory of partial differential equations. For example, fundamental solutions to partial differential equations are usually singular distributions and the behavior of the singularities of these distributions encodes the behavior of the solutions. Microlocal analysis deals with the detailed analysis of such distributions. It turns out that the singularities of distributions can be localized in phase space. This leads to the notion of wavefront sets, a refinement of the notion of singular support. Physicists may find this natural: the singularities of solutions to partial differential equations are described by geometrical optics, and geometrical optics is equivalent to a classical system on classical phase space. In fact, some ideas used in microlocal analysis were developed in less rigorous form by physicists. Asymptotic expansions, the WKB approximation, transport equations, all well familiar to physicists, appear also in microlocal analysis, sometimes a little bit disguised.

In these notes I will follow an approach which is to a large extent due to Hörmander [1–3] and which aims at a precise and rigorous treatment of the propagation of singularities of distributions. The notes are essentially self-contained but some knowledge of the basic notions of functional analysis and topology is assumed. The material is organized as follows. In Sect. 4.2 I recall the theory of distributions and of Fourier transforms of distributions on an elementary level. In Sect. 4.3 I focus on the concept of the wavefront set as a refinement of the notion of the singular support. Sections 4.3.1 and 4.3.2 follow closely the treatment of Hörmander's book

A. Strohmaier (✉)
Department of Mathematical Sciences, Loughborough University, Leicestershire LE11 3TU, UK
e-mail: A.Strohmaier@lboro.ac.uk

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[1, vol. I] and I discuss when a distribution may be pulled back under a smooth map and when two distributions may safely be multiplied. In Sect. 4.3.3 the wavefront sets of the fundamental solutions to the wave equation in Minkowski spacetime are calculated. In Sect. 4.3.4 the wavefront sets of the most commonly used scalar propagators of QFT in Minkowski spacetime are determined. Section 4.4 contains two fundamental results in microlocal analysis: microlocal elliptic regularity says in which direction solutions to partial differential equations may be singular and the propagation of singularity theorem clarifies how singularities of solutions to partial differential equations propagate. In Sect. 4.5 we demonstrate how these two results can be used to determine the wavefront sets of Green's distributions of any generalized d'Alembert operator on a globally hyperbolic spacetime. As realized by Radzikowski in his PhD thesis [4] the so-called Hadamard states, that are believed to be the physical states in QFT in curved spacetimes, can be characterized by their wavefront sets. I will give a brief explanation of this characterization in the end of these notes.

4.2 Distributions

4.2.1 Basic Definitions and Properties of Distributions

The so-called Dirac δ-function that is widely used in physics describes, for example, the density of a point mass: it is a positive function, has support at a single point, and it integrates to 1. It is an easy exercise to prove that there are no functions that satisfy the above properties. However, one can still calculate with this function and get reasonable results; and one can do this on a sound mathematical basis. It is the theory of distributions that achieves this.

In the following let $\mathcal{U}$ be a non-empty open subset of $\mathbb{R}^n$. The set of smooth complex-valued functions on $\mathcal{U}$ will be denoted by $\mathcal{E}(\mathcal{U}) = C^\infty(\mathcal{U})$ and the subset of functions with support compact in $\mathcal{U}$ will be denoted by $\mathcal{D}(\mathcal{U}) = C_0^\infty(\mathcal{U})$. Both spaces come equipped with topologies, namely $\mathcal{E}(\mathcal{U})$ is a Fréchet space with the family of semi-norms

$$p_{\alpha,K}(f) = \sup_{x \in K} |\partial^\alpha f(x)|. \tag{4.1}$$

where K runs over all compact subsets $K \subset \mathcal{U}$

This means a sequence of functions f_n converges to f in $\mathcal{E}(\mathcal{U})$ if and only if all its derivatives converge uniformly on compact subsets. Since any open set admits a countable exhaustion by compact subsets one can always replace the above family by an equivalent countable family of semi-norms. Thus, $\mathcal{E}(\mathcal{U})$ is indeed a Fréchet space.

The topology on $\mathcal{D}(\mathcal{U})$ is slightly more complicated to define. One chooses a compact exhaustion $K_1 \subset K_2 \subset, \ldots, \bigcup_n K_n = \mathcal{U}$. Then, let $\mathcal{D}(K_i)$ be the set of functions in $\mathcal{D}(\mathcal{U})$ with support in K_i. For each function in $\mathcal{D}(\mathcal{U})$ there exists an