Preliminaries on convex analysis and vector optimization

In this chapter we introduce some basic notions and results in convex analysis and vector optimization in order to make the book as self-contained as possible. The reader is supposed to have basic notions of functional analysis.

2.1 Convex sets

This section is dedicated mainly to the presentation of convex sets and their properties. With some exceptions the results we present in this section are given without proofs, as these can be found in the books and monographs on this topic mentioned in the bibliographical notes at the end of the chapter. All around this book we denote by $\mathbb{R}^n$ the $n$-dimensional real vector space, while by $\mathbb{R}^n_+ = \{ x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \ldots, n \}$ we denote its nonnegative orthant. By $\mathbb{N} = \{1, 2, \ldots\}$ we denote the set of natural numbers, while $\emptyset$ is the empty set. All the vectors are considered as column vectors. An upper index $^T$ transposes a column vector to a row one and vice versa. By $\mathbb{R}^{m \times n}$ we denote the space of the $m \times n$ matrices with real entries. When we have a matrix $A \in \mathbb{R}^{m \times n}$, by $A_i$, $i = 1, \ldots, m$, we denote its rows and, naturally, by $A^T$ its transpose. By $e^i \in \mathbb{R}^n$ we denote the $i$-th unit vector of $\mathbb{R}^n$, while by $e := \sum_{i=1}^n e^i \in \mathbb{R}^n$ we understand the vector having all entries equal to 1. If a function $f$ takes everywhere the value $a \in \mathbb{R}$ we write $f \equiv a$.

2.1.1 Algebraic properties of convex sets

Let $X$ be a real nontrivial vector space. A linear subspace of $X$ is a nonempty subset of it which is invariant with respect to the addition and the scalar multiplication on $X$. Note that an intersection of linear subspaces is itself a linear subspace. The algebraic dual space of $X$ is defined as the set of all linear functionals on $X$ and it is denoted by $X^\#$. Given any linear functional $x^\# \in X^\#$, we denote its value at $x \in X$ by $\langle x^\#, x \rangle$. 
For $x^\# \in X^\# \setminus \{0\}$ and $\lambda \in \mathbb{R}$ the set $\mathcal{H} := \{x \in X : \langle x^\#, x \rangle = \lambda \}$ is called hyperplane. The sets $\{x \in X : \langle x^\#, x \rangle \leq \lambda \}$ and $\{x \in X : \langle x^\#, x \rangle \geq \lambda \}$ are the closed halfspaces determined by the hyperplane $\mathcal{H}$, while $\{x \in X : \langle x^\#, x \rangle < \lambda \}$ and $\{x \in X : \langle x^\#, x \rangle > \lambda \}$ are the open halfspaces determined by $\mathcal{H}$. In order to simplify the presentation, the origins of all spaces will be denoted by 0, since the space where this notation is used always arises from the context. By $\Delta_{X^m}$ we denote the set $\{(x_1, \ldots, x_m) \in X^m : x \in X\}$, which is a linear subspace of $X^m := X \times \ldots \times X = \{(x_1, \ldots, x_m) : x_i \in X, i = 1, \ldots, m\}$.

If $U$ and $V$ are two subsets of $X$, their Minkowski sum is defined as $U + V := \{u + v : u \in U, v \in V\}$. For $U \subseteq X$ we define also $x + U = U + x := U + \{x\}$ when $x \in X$, $\lambda U := \{\lambda u : u \in U\}$ when $\lambda \in \mathbb{R}$ and $\Lambda U := \bigcup_{\lambda \in \Lambda} \lambda U$ when $\Lambda \subseteq \mathbb{R}$. According to these definitions one has that $U + \emptyset = \emptyset + U = \emptyset$ and $\lambda \emptyset = \emptyset$ whenever $U \subseteq X$ and $\lambda \in \mathbb{R}$. Moreover, if $U \subseteq V \subseteq X$ and $U \neq V$ we write $U \subsetneq V$.

Some important classes of subsets of a real vector space $X$ follow. Let be $U \subseteq X$. If $[-1, 1]U \subseteq U$, then $U$ is said to be a balanced set. When $U = -U$ we say that $U$ is symmetric, while $U$ is called absorbing if for all $x \in X$ there is some $\lambda > 0$ such that one has $x \in \lambda U$.

**Affine and convex sets.** Before introducing the notions of affine and convex sets, some necessary prerequisites follow. Taking some $x_i \in X$ and $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, n$, the sum $\sum_{i=1}^n \lambda_i x_i$ is said to be a linear combination of the vectors $\{x_i : i = 1, \ldots, n\}$. The vectors $x_i \in X$, $i = 1, \ldots, n$, are called linearly independent if from $\sum_{i=1}^n \lambda_i x_i = 0$ follows $\lambda_i = 0$ for all $i = 1, \ldots, n$. The linear hull of a set $U \subseteq X$,

$$ \text{lin}(U) := \left\{ \sum_{i=1}^n \lambda_i x_i : n \in \mathbb{N}, x_i \in U, \lambda_i \in \mathbb{R}, i = 1, \ldots, n \right\}, $$

is the intersection of all linear subspaces containing $U$, being the smallest linear subspace having $U$ as a subset.

The set $U \subseteq X$ is called affine if $\lambda x + (1 - \lambda)y \in U$ whenever $\lambda \in \mathbb{R}$. The intersection of arbitrarily many affine sets is affine, too. The smallest affine set containing $U$ or, equivalently, the intersection of all affine sets having $U$ as a subset is the affine hull of $U$,

$$ \text{aff}(U) := \left\{ \sum_{i=1}^n \lambda_i x_i : n \in \mathbb{N}, x_i \in U, \lambda_i \in \mathbb{R}, i = 1, \ldots, n, \sum_{i=1}^n \lambda_i = 1 \right\}. $$

A set $U \subseteq X$ is called convex if

$$ \{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\} \subseteq U \quad \text{for all } x, y \in U. $$

Obviously, $\emptyset$ and the whole space $X$ are convex sets, as well as the hyperplanes, linear subspaces, affine sets and any set containing a single element. An example of a convex set in $\mathbb{R}^n$ is the standard $(n - 1)$-simplex which is the set $\Delta_n := \{x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$. Given $x_i \in X$, $i = 1, \ldots, n$,