In this chapter we introduce different vector optimization problems corresponding vector dual problems by using some duality concepts having as starting point the scalar duality theory. Since there is a certain similarity between its definition and the one of the duals introduced via scalarization, we also investigate the classical geometric vector duality concept. More than that, we give a general approach for treating duality in vector optimization independently from the nature of the scalarization functions considered. We also provide a first look at the duality theory for linear vector optimization problems in Hausdorff locally convex spaces.

4.1 Fenchel type vector duality

Let $X, Y$ and $V$ be Hausdorff locally convex spaces and assume that $V$ is partially ordered by the nontrivial pointed convex cone $K \subseteq V$. Further, let $f : X \to V = V \cup \{ \pm \infty_K \}$ and $g : Y \to V$ be given proper and $K$-convex functions and $A \in L(X, Y)$ such that $\text{dom } f \cap A^{-1}(\text{dom } g) \neq \emptyset$.

To the primal vector optimization problem

$$(PV^A) \ \text{Min}_{x \in X} \{ f(x) + g(Ax) \}$$

we introduce dual vector optimization problems with respect to both properly efficient solutions in the sense of linear scalarization and weakly efficient solutions and prove weak, strong and converse duality theorems.

4.1.1 Duality with respect to properly efficient solutions

In this subsection we investigate a duality approach to $(PV^A)$ with respect to properly efficient solutions in the sense of linear scalarization. Since we do not have to differentiate between different classes of such solutions we call them simply properly efficient solutions. We say that $\bar{x} \in X$ is a properly efficient
solution to $(PV^A)$ if $\bar{x} \in \text{dom } f \cap A^{-1}(\text{dom } g)$ and $f(\bar{x}) \in \text{PMin}_{LSC}((f + g \circ A)(\text{dom } f \cap A^{-1}(\text{dom } g)), K)$. This means that there exists $v^* \in K^{*0}$ such that $\langle v^*, (f + g \circ A)(\bar{x}) \rangle \leq \langle v^*, (f + g \circ A)(x) \rangle$ for all $x \in X$.

The vector dual problem to $(PV^A)$ we investigate in this subsection is

$$(DV^A) \quad \max_{(v^*, y^*, v) \in B^A} h^A(v^*, y^*, v),$$

where

$$B^A = \{(v^*, y^*, v) \in K^{*0} \times Y^* \times V : \langle v^*, v \rangle \leq -(v^* f)(-A^* y^*) - (v^* g)(y^*)\}$$

and

$$h^A(v^*, y^*, v) = v.$$

We prove first the existence of weak duality for $(PV^A)$ and $(DV^A)$.

**Theorem 4.1.1.** There is no $x \in X$ and no $(v^*, y^*, v) \in B^A$ such that $(f + g \circ A)(x) \leq_K h^A(v^*, y^*, v)$.

**Proof.** We assume the contrary, namely that there exist $x \in X$ and $(v^*, y^*, v) \in B^A$ such that $v - (f + g \circ A)(x) = h^A(v^*, y^*, v) - (f + g \circ A)(x) \geq_K 0$. It is obvious that $x \in \text{dom } f \cap A^{-1}(\text{dom } g)$ and that $\langle v^*, v \rangle > \langle v^*, f(x) \rangle + \langle v^*, g(Ax) \rangle$.

On the other hand, we have

$$\langle v^*, f(x) \rangle + \langle v^*, g(Ax) \rangle \geq \inf_{y \in X} \{\langle v^*, f(y) \rangle + \langle v^*, g(Ay) \rangle\}.$$

Since the infimum on the right-hand side of the relation above is greater than or equal to the optimal objective value of its corresponding scalar Fenchel dual problem (cf. subsection 3.1.2), it holds

$$\langle v^*, v \rangle > \inf_{y \in X} \{(v^* f)(y) + (v^* g)(Ay)\}$$

$$\geq \sup_{z^* \in Y^*} \{- (v^* f)^*(-A^* z^*) - (v^* g)^*(z^*)\} \geq - (v^* f)^*(-A^* y^*) - (v^* g)^*(y^*).$$

As this contradicts the fact that $(v^*, y^*, v) \in B^A$, the conclusion follows. \(\square\)

In order to be able to prove the following strong duality theorem for the primal-dual vector pair $(PV^A) - (DV^A)$ we have to impose a regularity condition which actually ensures the existence of strong duality for the scalar problem

$$\inf_{x \in X} \{(v^* f)(x) + (v^* g)(Ax)\}$$

and its Fenchel dual

$$\sup_{y^* \in Y^*} \{- (v^* f)^*(-A^* y^*) - (v^* g)^*(y^*)\},$$

for all $v^* \in K^{*0}$. This means that we are looking for sufficient conditions which are independent from the choice of $v^* \in K^{*0}$ and therefore we consider the following regularity condition.