Wolfe and Mond-Weir duality concepts

In this chapter we present scalar and vector duality based on the classical Wolfe and Mond-Weir duality concepts. As the field is very vast, especially because of different generalizations of the notion of convexity for the functions employed, we limited our exposition to a reasonable framework, large enough to present the most relevant facts in the area. We shall work in parallel with the two mentioned duality concepts. Note that the properly efficient solutions that appear in this chapter are considered in the sense of linear scalarization (see Definition 2.4.12), unless otherwise specified.

6.1 Classical scalar Wolfe and Mond-Weir duality

For the beginnings of Wolfe duality the reader is referred to [202], while the first paper on Mond-Weir duality is considered to be [138]. In both of them the functions involved were taken differentiable, but afterwards both these duality concepts were extended to nondifferentiable functions by making use of convexity. We begin our presentation with the convex case, treating after that the situation when the functions involved are assumed moreover differentiable.

6.1.1 Scalar Wolfe and Mond-Weir duality: nondifferentiable case

Like in section 3.1.3, let $X$ and $Z$ be Hausdorff locally convex spaces, the latter partially ordered by the convex cone $C \subseteq Z$, and consider the nonempty set $S \subseteq X$ and the proper functions $f : X \to \mathbb{R}$ and $g : X \to Z$, fulfilling $\text{dom} f \cap S \cap g^{-1}(-C) \neq \emptyset$. The primal problem we treat further is

$$\left( P^C \right) \inf_{x \in A} f(x),$$

$A = \{x \in S : g(x) \in -C\}$

To it we attach the Wolfe dual problem
Theorem 6.1.1. One has \( v(D^C_{MW}) \leq v(D^C_W) \leq v(PC) \).

Proof. We distinguish two cases. If the feasible set of \((D^C_W)\) is empty, then so is the one of \((D^C_{MW})\), in which case the optimal objective values of these problems are both equal to \(-\infty\), which is clearly less than or equal to \(v(PC)\).

Otherwise, let \(u \in S\) and \(z^* \in C^*\), fulfilling \(0 \in \partial f(u) + \partial (z^* g)(u) + N(S, u)\). If \(\langle z^*, g(u) \rangle \geq 0\) then \((u, z^*)\) is feasible to \((D^C_{MW})\) and \(f(u) \leq f(u) + \langle z^*, g(u) \rangle\). Taking now in both sides of this inequality the suprema regarding all pairs \((u, z^*)\) feasible to \((D^C_{MW})\) we obtain in the left-hand side \(v(D^C_{MW})\), while in the right-hand side there is the supremum of the objective function of \((D^C_W)\) concerning only some of the feasible solutions to this problem. Consequently, \(v(D^C_{MW}) \leq v(D^C_W)\).

Since \(0 \in \partial f(u) + \partial (z^* g)(u) + N(S, u)\), by (2.8) follows \(0 \in \partial (f + (z^* g) + \delta_S)(u)\), i.e. for all \(x \in S\) one has \(f(x) + \langle z^*, g(x) \rangle \geq f(u) + \langle z^*, g(u) \rangle\). Taking in the left-hand side of this inequality the infimum regarding all \(x \in S\) for which \(g(x) \in -C\), we obtain there a value less than \(v(PC)\). Considering then in the right-hand side of the same inequality the supremum regarding all pairs \((u, z^*)\) feasible to \((D^C_W)\), it follows \(v(D^C_W) \leq v(PC)\). \(\square\)

Theorem 6.1.2. Assume that \(S \subseteq X\) is a convex set, \(f : X \to \overline{\mathbb{R}}\) a proper and convex function and \(g : X \to \overline{\mathbb{R}}\) a proper and \(C\)-convex function such that \(\text{dom } f \cap S \cap g^{-1}(-C) \neq \emptyset\). If one of the regularity conditions \((RC^C_i)\), \(i \in \{1, 2, 3, 4\}\) is fulfilled, \((PC)\) has an optimal solution \(\bar{x} \in A\) and one of the following additional conditions

(i) \(f\) and \(g\) are continuous at a point in \(\text{dom } f \cap \text{dom } g \cap S\);
(ii) \(\text{dom } f \cap \text{dom } g \cap \text{int}(S) \neq \emptyset\) and \(f\) or \(g\) is continuous at a point in \(\text{dom } f \cap \text{dom } g\);
(iii) \(X\) is a Fréchet space, \(f\) is lower semicontinuous, \(g\) is star \(C\)-lower semicontinuous, \(S\) is closed and \(0 \in \text{sqri}(\text{dom } f \times \text{dom } g \times S - \Delta_{X^2})\);
(iv) \(\text{dim}(\text{lin}(\text{dom } f \times \text{dom } g \times S - \Delta_{X^3})) < +\infty\) and \(0 \in \text{ri}(\text{dom } f \times \text{dom } g \times S - \Delta_{X^3})\);

is fulfilled, then \(v(PC) = v(D^C_W) = v(D^C_{MW})\) and there is some \(z^* \in C^*\) with \((\bar{x}, z^*)\) is an optimal solution to each of the duals.