Chapter 6
The Continuous Averaging Method

6.1 Description of the Method

There are several problems in perturbation theory, where standard methods do not lead to satisfactory results. We mention as examples the problem of an inclusion of a diffeomorphism into a flow in the analytic set up, and the problem of quantitative description of exponentially small effects in dynamical systems. In these cases one possible approach is based on the continuous averaging. The method appeared as an extension of the Neishtadt averaging procedure [95], effectively working in the presence of exponentially small effects.

To present the general idea let us transform the system of ordinary differential equations

\[ \dot{z} = \hat{u}(z), \]  

by using the change of variables

\[ z \mapsto Z(z, \Delta). \]  

Here \( z \) is a point of the manifold \( M \), \( \hat{u} \) is a smooth vector field on \( M \), \( \Delta \) is a non-negative parameter, and change (6.2) is defined as a shift along solutions of the equation

\[ Z' = f(Z, \delta), \quad Z(z, 0) = z, \quad 0 \leq \delta \leq \Delta, \]  

where the prime denotes the derivative with respect to \( \delta \).

Let the change \( z \mapsto Z \) transform (6.1) to the following system:

\[ \dot{Z} = u(Z, \delta). \]  

Differentiating equation (6.4) with respect to \( \delta \), we have

\[ \dot{f}(Z, \delta) = u_\delta(Z, \delta) + \partial_f u(Z, \delta) \quad \text{or} \quad u_\delta = [u, f]. \]

---

1 Such a construction for a change of variables is called the Lie method. The corresponding Hamiltonian version is called the Deprit–Hori method.
Here $\partial_f$ is the differential operator on $M$ corresponding to the vector field $f$, the subscript $\delta$ denotes the partial derivative, and $[\cdot, \cdot]$ is the vector commutator: $[u_1, u_2] = \partial_{u_1} u_2 - \partial_{u_2} u_1$. Putting $f = \xi u$, where $\xi$ is some fixed linear operator, we obtain the Cauchy problem

$$u_\delta = -[\xi u, u], \quad u|_{\delta=0} = \hat{u}. \quad (6.5)$$

The equation $f = \xi u$ is crucial for our method. Traditionally the vector field $f$ in the Lie method is constructed as a series in the small parameter. The choice of the operator $\xi$ depends on the form to which we want to transform the initial equations. We call (6.5) an averaging system.

If (6.1) is a Hamiltonian system with Hamiltonian $\hat{H} = \hat{H}(z)$ and the symplectic structure $\omega$, it is natural to search for the change (6.2) among symplectic ones, and to regard equation (6.3) as Hamiltonian with some Hamiltonian function $F(z, \delta)$. Under these assumptions systems (6.4) are also Hamiltonian. Their Hamiltonian functions $H$ satisfy the equation $H(Z, \delta) = \hat{H}(z)$. Differentiating this equation with respect to $\delta$, we get

$$H_\delta(Z, \delta) + \partial_f(Z, \delta) H(Z, \delta) = 0 \quad \text{or} \quad H_\delta = -\{F, H\}$$

because $\partial_f H = \{F, H\}$. Putting $F = \xi H$ for some linear operator $\xi$, we obtain

$$H_\delta = -\{\xi H, H\}, \quad H|_{\delta=0} = \hat{H}. \quad (6.6)$$

Now let us present a nonautonomous analog of (6.6). To obtain such an analog, assume that the functions $\hat{H}$ and $F$ depend explicitly on time. Then the Hamiltonian $H$ also depends explicitly on $t$. We obtain an equation for $H$ by the reduction to the autonomous case. Let $E$ be a variable, canonically conjugate to time $t$. Consider the autonomous system with Hamiltonian $H + E$ and the symplectic structure $\omega + dE \wedge dt$. Let $\{\cdot, \cdot\}_*$ be the new Poisson bracket and $F = \xi H$. Then (6.6) takes the form:

$$(H + E)_\delta = -\{\xi H, H + E\}_*, \quad H|_{\delta=0} = \hat{H}. \quad (6.7)$$

It is equivalent to the following one:

$$H_\delta = (\xi H)_t - \{\xi H, H\}, \quad H|_{\delta=0} = \hat{H}(z, t). \quad (6.7)$$

This is a nonautonomous analog of system (6.6).

Analogously a nonautonomous analog of (6.5) can be constructed:

$$u_\delta = (\xi u)_t - [\xi u, u], \quad u|_{\delta=0} = \hat{u}(z, t). \quad (6.8)$$

Properties of the averaging system can be illustrated by the following example. Consider the Hamiltonian system with one and a half degrees of freedom

$$\dot{y} = -\varepsilon \partial \hat{H} / \partial x, \quad \dot{x} = \varepsilon \partial \hat{H} / \partial y, \quad \hat{H} = \hat{H}(x, y, t). \quad (6.9)$$