Chapter 7
The Anti-Integrable Limit

7.1 Perturbation of the Standard Map

We have seen in the previous chapters that the dynamics in near-integrable Hamiltonian systems and symplectic maps remains quite regular: stochastic regimes exist, but are located in small domains. It is natural to expect that chaotic properties become more pronounced when the “distance to the set of integrable systems” increases.

Consider as an example the standard map (1.5) from Chap. 1, where the parameter \( \varepsilon \) is not small but, on the contrary, very large. The limit as \( \varepsilon \to \infty \) in systems of such a type is called the anti-integrable limit [12].

Let us rewrite the map \( SM \) in the “Lagrangian form”. To this end suppose that

\[
\begin{pmatrix}
  x_- \\
y_-
\end{pmatrix}
\mapsto
\begin{pmatrix}
x \\
y
\end{pmatrix}
\mapsto
\begin{pmatrix}
x_+ \\
y_+
\end{pmatrix}
\]

Then \( x_-, x, x_+ \) satisfy the equation

\[
\varepsilon^{-1}(x_+ - 2x + x_-) = \sin x. \tag{7.1}
\]

The standard map written in this form is defined on the cylinder \( \mathcal{C} = \mathbb{R}^2_{(x-,x)}/\sim \), where the equivalence relation \( \sim \) is as follows:

\[
(x'_1, x'_2) \sim (x''_1, x''_2) \quad \text{if and only if } x'_1 - x''_1 = x'_2 - x''_2 \in 2\pi\mathbb{Z}.
\]

In the other words, the cylinder \( \mathcal{C} \) is the quotient space of the plane \( \mathbb{R}^2_{(x-,x)} \) with respect to the action of the group of shifts

\[
(x_-, x) \mapsto (x_- + 2\pi l, x + 2\pi l), \quad l \in \mathbb{Z}.
\]

The standard map sends the point \( (x_-, x) \in \mathcal{C} \) to \( (x, x_+) \in \mathcal{C} \), where \( x_-, x, x_+ \) satisfy (7.1).
Infinite sequences \(\ldots, x_{-1}, x_0, x_1, \ldots\) such that the triple \((x_{-1}, x, x_+)) = (x_{l-1}, x_l, x_{l+1})\) satisfies (7.1) for any integer \(l\), are called trajectories of the standard map.

The Lagrangian form of the standard map admits a variational formulation. Namely, trajectories of the system are extremals of the formal sum

\[
\sum_{l=-\infty}^{\infty} L(x_l, x_{l+1}), \quad L(x', x'') = \frac{1}{2\varepsilon} (x' - x'')^2 - \cos x''.
\] (7.2)

The extremality means that, for any trajectory \(\ldots, x_{0}^{0}, x_{0}^{0}, x_{1}^{0}, \ldots\) and any integer \(n\),

\[
\frac{\partial}{\partial x_n} \sum_{l=-\infty}^{\infty} L(x_l, x_{l+1}) = 0 \quad \text{for} \quad \ldots, x_{-1}, x_0, x_1, \ldots = \ldots, x_{0}^{0}, x_{0}^{0}, x_{1}^{0}, \ldots
\]

If \(\varepsilon = \infty\), the standard map is meaningless because \(x_+\) cannot be found in terms of \(x\) and \(x_-\) from equation (7.1)\(|_{\varepsilon = 1} = 0\). However, the corresponding variational problem is well-defined. Its solutions are sequences of the form

\[
\ldots, \pi k_{-1}, \pi k_0, \pi k_1, \ldots, \quad k_j \in \mathbb{Z}.
\] (7.3)

For large values of the parameter \(\varepsilon\) the standard map has many trajectories close to sequences (7.3). More precisely the following Theorem 7.1 holds [12]. Let \(S_K\) be the set of all sequences (7.3) such that \(|k_l - k_{l+1}| \leq K\) for any \(l \in \mathbb{Z}\).

Let us introduce on the set of infinite sequences the metric \(\rho\), corresponding to the uniform convergence norm, i.e., for any two sequences

\[
X' = \ldots, x'_{-1}, x'_0, x'_1, \ldots, \quad X'' = \ldots, x''_{-1}, x''_0, x''_1, \ldots
\]

we put

\[
\hat{\rho}(X', X'') = \sup_{l \in \mathbb{Z}} |x'_l - x''_l|.
\]

(For some pairs \(X', X''\) the last expression can be equal to infinity).

**Theorem 7.1 ([12]).** For any \(K > 0\) and any \(\sigma > 0\) there exists a (sufficiently large) \(\varepsilon_0 > 0\) such that, for any \(X' \in S_K\) and any \(\varepsilon > \varepsilon_0\), the standard map has a unique trajectory \(X''\) with \(\rho(X', X'') < \sigma\).

Theorem 7.1 means that for large values of \(\varepsilon\) some trajectories of the standard map turn out to be in a one-to-one correspondence with elements of the set \(S_K\). Sequences from \(S_K\) can be regarded as codes of the corresponding trajectories. This possibility to code trajectories by elements of a sufficiently large set is typical for chaotic systems.

Theorem 7.1 was generalized by MacKay and Meiss [83] to multidimensional symplectic maps with configurational space \(\mathbb{T}^m\). Below we present a proof of a theorem on the anti-integrable limit in a more general situation.