Budgets of polymer free energy in homogeneous turbulence

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Turbulence in dilute polymer solutions has gained more and more interest over the last decades. Original studies were mainly oriented to drag reduction observed in wall bounded flows, however, recently studies have moved from applied problems to a more fundamental approach. In details a renewed interest has been devoted to problems different from bounded flows such as jets or basically homogeneous flows. On the other hand from a theoretical viewpoint some relevant development have appeared. In this contribution the analysis of spectral budgets of homogeneous and isotropic turbulence based on the set of equations derived in [1] is proposed. The study is performed on low-Reynolds number numerical results obtained for a dilute polymer solution in the mild stretch regime.

1 Mathematical formulation and results

Under mild stretching, the dynamics of the ensemble of polymers is described by the linear and homogeneous equation for the conformation tensor, $\mathbf{R}$, [1]

$$\frac{\partial \mathbf{R}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{R} = \mathbf{K} \mathbf{R} + \mathbf{R} \mathbf{K}^\dagger - \frac{2}{\tau} \mathbf{R},$$

accounting for advection, stretching, re-orientation and linear elastic restoring force. As follows from its physical meaning, the conformation tensor must be a symmetric positive definite second order tensor. It can be factorized in terms of $\mathbf{X}$, the matrix of the right-eigenvectors, and $\Lambda$, the diagonal matrix of the eigenvalues, as $\mathbf{R} = \mathbf{X} \Lambda \mathbf{X}^\dagger$. As such, its square root, i.e. the tensor $\mathbf{Q}$ such that $\mathbf{R} = \mathbf{Q} \mathbf{Q}^\dagger$ with $\mathbf{Q} = \mathbf{X} \sqrt{\Lambda}$ exists and obeys the evolution equation

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{Q} = \mathbf{K} \mathbf{Q} - \frac{1}{\tau} \mathbf{Q}.$$
It is straightforward to show that $\nabla \cdot {\mathbf{Q}^\dagger}$ (in Cartesian components $\partial Q_{ji}/\partial x_j$) tends exponentially to zero with a time constant given by $\tau$. On the contrary, $\nabla \cdot {\mathbf{Q}}$ (in Cartesian components $\partial Q_{ij}/\partial x_j$) is not zero and is not conserved by the evolution implied by equation (2). In this framework, the elastic energy can be expressed as a quadratic form in terms of $\mathbf{Q}$, $\mathcal{E}_p(\mathbf{x}, t) := \nu_p/\tau \text{tr} \left[ \mathbf{Q}(\mathbf{x}, t) \mathbf{Q}^\dagger(\mathbf{x}, t) \right]$ while the production term can be written as $\Pi_p(\mathbf{x}, t) = 2\nu_p/\tau \text{tr} \left[ \mathbf{K}(\mathbf{x}, t) \mathbf{Q}(\mathbf{x}, t) \mathbf{Q}^\dagger(\mathbf{x}, t) \right]$. In other words the adoption of $\mathbf{Q}$ as descriptor for the polymers allows the physical energy to be expressed as the natural $L^2$-norm of the relevant field hence under the assumption of homogeneity, the averaged energy equation for the polymers takes the form

$$\frac{d\langle\mathcal{E}_p\rangle(t)}{dt} = \langle\Pi_p\rangle(t) - \frac{2}{\tau}\langle\mathcal{E}_p\rangle(t) \quad (3)$$

For the velocity, the average kinetic energy density, $\langle\mathcal{E}_k\rangle(t) = 1/2 \langle \mathbf{u} \cdot \mathbf{u} \rangle(t)$, follows a balance equation which for homogeneous fields reduces to

$$\frac{d\langle\mathcal{E}_k\rangle(t)}{dt} = \langle W\rangle(t) - \langle \epsilon_N \rangle(t) - \langle \Pi_N \rangle(t) \quad (4)$$

Homogeneous and isotropic turbulence has been simulated via a spectral code on $96^3$ grid points for the dealiasing procedure and the time history of the two energy components is shown in the left panel of figure 1. The corresponding Newtonian simulation, i.e. with equivalent viscosity and energy input, was characterised by a $Re_\lambda = 80$ and the Deborah number based on the Newtonian Kolmogorov time scale is equal to 5.

The $L^2$ formulation previously discussed entails the spectral decomposition of the elastic energy. Actually, for a homogeneous field, the three-dimensional spectrum of elastic energy may be defined from the correlation tensor of the field $\mathbf{Q}$, $\mathbf{C}_p(\mathbf{r}, t) := \langle \mathbf{Q}(\mathbf{x}, t) \mathbf{Q}^\dagger(\mathbf{x} + \mathbf{r}, t) \rangle$, as

$$E_p^{(3D)}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \frac{\nu_p}{\tau} \int_{\mathbb{R}^3} \text{tr} \left[ \mathbf{C}_p(\mathbf{r}, t) \right] e^{i\mathbf{k} \cdot \mathbf{r}} d^3\mathbf{r} \quad (5)$$