Chapter 2
Input/Output Map and Additive Noises

To obtain concrete results, we will consider a case of dynamical systems with an input/output mechanism surrounded by free noise or noise.

2.1 Input Response Maps (Input/Output Maps with Causality)

We will consider a notational method for input/output relations of an object to be observed or to be controlled in a discrete-time case, i.e., a black-box to which any element of the concatenation monoid $U^*$ can be applied and whose output values are in a set of output values, where $U^*$ is the free monoid over the input value’s set $U$. Sometimes, $\Omega$ may be used in place of $U^*$, namely $\Omega = U^*$ always holds. $Y$ is the set of output values. The representation theorems for any input/output map with causality have been given by [Matsuo and Hasegawa 2003]. The theorems can be stated as Lemmas (2.1), (2.5) and (2.8).

Lemma 2.1. Any input/output relation with causality can be represented as $a \in F(U^*, Y)$. Then, any $a \in F(U^*, Y)$ can be represented as the following equation:

$$\hat{\gamma}(|\omega|) = a(\omega) \in Y,$$

where $\hat{\gamma}(|\omega|)$ denotes an output value at the time $|\omega|$ for an input $\omega$ to have been ended to apply, where $|\omega|$ is the length of the input $\omega$.

Definition 2.2. An element $a$ of $F(U^*, Y)$ is said to be an input response map.

For the convenience of our discussions, we have utilized some kinds of input response maps from [Matsuo and Hasegawa, 2003].

Definition 2.3. If an input response map $a \in F(U^*, Y)$ satisfies the following time-invariant condition, then $a$ is said to be a time-invariant input response map.
Time-invariant condition: \(a(\omega_1|\omega) - a(\omega_1) = a(\bar{\omega}_1|\omega) - a(\bar{\omega}_1)\) for any \(\omega \in U^*\), and \(\omega_1, \bar{\omega}_1 \in U^*\) such that \(|\omega_1| = |\bar{\omega}_1|\).

**Definition 2.4.** For any time-invariant input response map \(a \in F(U^*, Y)\), a function \(I_a : U \to F(U^*, Y); u \mapsto I_a(u); t \mapsto a(u^t) - a(u^{t-1})\) is said to be a modified impulse response of \(a\), where \(u^t\) is given by \(u^t(i) = u\) for \(i(1 \leq i \leq t)\).

**Lemma 2.5.** Representation Theorem
For any time-invariant input response map \(a \in F(U^*, Y)\), there exist uniquely modified impulse responses represented by the following equation. This correspondence is bijective.

\[
a(\omega) = a(1) + \sum_{j=1}^{|\omega|} (I_a(\omega(j)))(|\omega| - j).
\]

In our case, we consider input/output maps \(a \in F(U^*, Y)\) which satisfy the following time-invariant condition and affinity condition. They are said to be time-invariant, affine input response maps, where \(U\) is a linear space in this case. We may treat the case where multi-inputs are fed, i.e., \(U = \mathbb{R}^m\), but conveniently, we will discuss a case where one-input is fed, i.e., \(U = \mathbb{R}\). And \(Y\) is a linear space over the real number field \(\mathbb{R}\).

**Definition 2.6.** If an input response map \(a\) satisfies the following time-invariant and affinity condition, then \(a\) is said to be a time-invariant, affine input response map.

Time-invariant condition:
\[
a(\omega_1|\omega) - a(\omega_1) = a(\bar{\omega}_1|\omega) - a(\bar{\omega}_1)
\]
for any \(\omega, \omega_1, \bar{\omega}_1\) such that \(|\omega_1| = |\bar{\omega}_1|\).

Affinity condition:
\[
a(\omega + \bar{\omega}) + a(0|\omega) = a(\omega) + a(\bar{\omega})
\]
\[
a(\lambda \omega) = \lambda a(\omega) + (1 - \lambda) a(0|\omega)
\]
for any \(\omega, \bar{\omega} \in U^*, |\omega| = |\bar{\omega}|\) and \(\lambda \in K\).

**Definition 2.7.** For any time-invariant, affine input response map \(a \in F(U^*, Y)\), a function \(I_a : \{0, 1\} \to F(N, Y); u \mapsto I_a(u); t \mapsto a(u^t) - a(u^{t-1})\) is said to be a modified impulse response of \(a\).

**Lemma 2.8.** Representation Theorem
For any time-invariant, affine input response map \(a \in F(U^*, Y)\), there exist uniquely modified impulse responses represented by the following equation. This correspondence is bijective.

\[
a(\omega) = a(1) + \sum_{j=1}^{|\omega|} (I_a(\omega(j))) (I_a(1)(|\omega| - j + 1)) + (1 - \omega(j))(I_a(0)(|\omega| - j + 1))
\]
for any \(\omega \in U^*\).

The problem of approximation and noisy realization for input/output relation with causality are roughly stated as follows:

**Problem 2.9.** Problem statement for approximate realization
For any given data of the input/output map, find an input response map which is suitable in the sense of approximation, namely, a dynamical system with approximate behavior for the given input response map.