Chapter 4
Algebraically Approximate and Noisy Realization of So-called Linear Systems

Let the set $Y$ of output’s values be a linear space over the real number field $\mathbb{R}$.

Almost linear systems were introduced in the monograph [Matsuo and Hasegawa, 2003], and it was also shown that the systems contain so-called linear systems as a sub-class, where so-called linear systems are linear systems with a non-zero initial state.

It is well known that a common method to obtain so-called linear systems is solved through two problems.

One is the realization problem to obtain linear systems with a zero initial state and the other is the state estimation problem for systems with a non-zero initial state. Based upon the prejudice that so-called linear systems are completely the same as linear systems, so-called linear systems were treated separately.

In the monograph, it was also shown that so-called linear systems can be obtained from input/output data from a single experiment.

In this chapter, based on the results regarding so-called linear systems, we will discuss algebraically approximate and noisy realization of the systems. For our discussion, we will present for the first time a concrete and easy method to discuss algebraically approximate and noisy realization problems from partial data, equivalently, i.e., data obtained in finite real time. Hence, this new method is very useful and practical.

Note that because of the system’s nonlinearity, these problems were discussed by using the analytic CLS method the first time in the reference [Hasegawa, 2008].

In order to be self-contained, we will list the main results needed for our discussion from our monograph.

In order to solve our problems, we will use singular value decomposition and the algebraically Constrained Least Square method, which is abbreviated to the algebraic CLS method which has been discussed in Chapter 2. The singular value decomposition is used to determine the dimension of so-called
linear systems and the algebraic CLS method is used to determine parameters of a so-called linear system.

At first, we will discuss algebraically approximate realization problems and give many examples to ascertain the effectiveness of our algorithm. Next, we will discuss algebraically noisy realization problems and give several examples to ascertain the effectiveness of our algebraically noisy realization algorithm. Both an approximate realization problem and a noisy realization problem will be discussed through executing only algebraic operations in comparison with the analytic CLS method in the reference [Hasegawa, 2008].

4.1 Basic Facts about So-called Linear Systems

Definition 4.1. So-called Linear Systems

1) A system given by the following system equation is said to be a so-called linear system $\sigma = ((X, F), x^0, g, h)$. This system is a linear system with a non-zero initial state.

$$\begin{align*}
x(t + 1) &= Fx(t) + g\omega(t + 1) \\
x(0) &= x^0 \\
\gamma(t) &= hx(t)
\end{align*}$$

where $F \in L(X)$, $\omega(t + 1) \in U$, $g$, $x^0 \in X$. In addition, $h$ is a linear operator : $X \rightarrow Y$ for any $t \in N$, $\gamma(t) \in Y$.

2) The input response map $a_\sigma : U^* \rightarrow Y; \omega \mapsto h(\sum_{|\omega|} F^{|\omega|} - 1 (Fx^0 + g\omega(j)))$ is said to be the behavior of $\sigma$.

3) For the so-called linear system $\sigma$ and any $i \geq 1$,

$I_\sigma(1)(i) := a_\sigma(0^i|1) - a_\sigma(0^i) = hF^i(g^0 + g)$ and

$I_\sigma(0)(i) := a_\sigma(0^i+1) - a_\sigma(0^i) = hF^ig^0$ are said to be modified impulse responses of $\sigma$, where $0^0 := 1$, $g^0 := Fx^0 - x^0$.

Note that there is a one-to-one correspondence between the behavior of $\sigma$ and the modified impulse responses $I_\sigma(0)$ and $I_\sigma(1) \in F(N, Y)$ of $\sigma$ by the relations $a_\sigma(\omega) = (\sum_{|\omega|} I_\sigma(0)(|\omega| - j + 1) + I_\sigma(1)(|\omega| - j + 1) \times \omega(j))$.

4) A so-called linear system $\sigma$ is said to be reachable if the reachable set

$$\{\sum_{j=1}^{|\omega|} F^{|\omega|} - j (g^0 + g\omega(j)); \omega \in U^*\}$$

is equal to $X$ and the system $\sigma$ is called to be observable if $hF^i x_1 = hF^i x_2$ for any $i \in N$ implies $x_1 = x_2$, where $g^0 := Fx^0 - x^0$.

5) A so-called linear system $\sigma$ is called canonical if $\sigma$ is reachable and observable.

Remark 1: It is meant for $\sigma$ to be a faithful model for the input response map $a$ that $\sigma$ realizes $a$.

Remark 2: Notice that a canonical so-called linear system $\sigma = ((X, F), x^0, g, h)$ is a system that has the most reduced state space $X$ among systems that have the behavior $a_\sigma$. 