On Reconfiguration of Disks in the Plane and Related Problems

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Abstract. We revisit two natural reconfiguration models for systems of disjoint objects in the plane: translation and sliding. Consider a set of \( n \) pairwise interior-disjoint objects in the plane that need to be brought from a given start (initial) configuration \( S \) into a desired goal (target) configuration \( T \), without causing collisions. In the translation model, in one move an object is translated along a fixed direction to another position in the plane. In the sliding model, one move is sliding an object to another location in the plane by means of an arbitrarily complex continuous motion (that could involve rotations). We obtain several combinatorial and computational results for these two models:

(I) For systems of \( n \) congruent disks in the translation model, Abellanas et al. \cite{Abellanas2009} showed that \( 2^n - 1 \) moves always suffice and \( \lfloor \frac{8n}{5} \rfloor \) moves are sometimes necessary for transforming the start configuration into the target configuration. Here we further improve the lower bound to \( \lfloor \frac{5n}{3} \rfloor - 1 \), and thereby give a partial answer to one of their open problems.

(II) We show that the reconfiguration problem with congruent disks in the translation model is NP-hard, in both the labeled and unlabeled variants. This answers another open problem of Abellanas et al. \cite{Abellanas2009}.

(III) We also show that the reconfiguration problem with congruent disks in the sliding model is NP-hard, in both the labeled and unlabeled variants.

(IV) For the reconfiguration with translations of \( n \) arbitrary convex bodies in the plane, \( 2^n \) moves are always sufficient and sometimes necessary.

1 Introduction

A body (or object) in the plane is a compact connected set in \( \mathbb{R}^2 \) with nonempty interior. Two initially disjoint bodies collide if they share an interior point at some time during their motion. Consider a set of \( n \) pairwise interior-disjoint

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objects in the plane that need to be brought from a given start (initial) configuration \( S \) into a desired target (goal) configuration \( T \), without causing collisions. The reconfiguration problem for such a system is that of computing a sequence of object motions (a schedule, or motion plan) that achieves this task. Depending on the existence of such a sequence of motions, we call that instance of the problem feasible and respectively, infeasible.

Our reconfiguration problem is a simplified version of the multi-robot motion planning problem, in which a system of robots are operating together in a shared workplace and once in a while need to move from their initial positions to a set of target positions. The workspace is often assumed to extend throughout the entire plane, and has no obstacles other than the robots themselves. In many applications, the robots are indistinguishable (unlabeled), so each of them can occupy any of the specified target positions. Beside multi-robot motion planning, another application which permits the same abstraction is moving around large sets of heavy objects in a warehouse \([10]\). Typically, one is interested in minimizing the number of moves and designing efficient algorithms for carrying out the motion plan. It turns out that moving a set of objects from one place to another is related to certain separability problems \([4,6,7,9]\); see also \([11]\). There are several types of moves that make sense to study, as dictated by specific applications. Here we focus on systems of convex bodies in the plane.

Next we formulate these models for systems of disks, since they are simpler and most of our results are for disks. These rules can be extended (not necessarily uniquely) for arbitrary convex bodies in the plane. The decision problems we refer to below, pertaining to various reconfiguration problems we discuss here, are in standard form, and concern systems of (arbitrary or congruent) disks. For instance, the Reconfiguration Problem U-SLIDE-RP for congruent disks is: Given a start configuration and a target configuration, each with \( n \) unlabeled congruent disks in the plane, and a positive integer \( k \), is there a reconfiguration motion plan with at most \( k \) sliding moves? It is worth clarifying that for the unlabeled variant, if the start and target configuration contain subsets of congruent disks, there is freedom is choosing which disks will occupy target positions. However in the labeled variant, this assignment is uniquely determined by the labeling; of course a valid labeling must respect the size of the disks.

1. **Sliding model**: one move is sliding a disk to another location in the plane without colliding with any other disk, where the disk center moves along an arbitrary continuous curve. This model was introduced in \([3]\). The labeled and unlabeled variants are L-SLIDE-RP and U-SLIDE-RP, respectively.

2. **Translation model**: one move is translating a disk to another location in the plane along a fixed direction without colliding with any other disk. This is a restriction imposed to the sliding model above for making each move as simple as possible. This model was introduced in \([1]\). The labeled and unlabeled variants are L-TRANS-RP and U-TRANS-RP, respectively.

3. **Lifting model**: one move is lifting a disk and placing it back in the plane anywhere in the free space. This model was introduced in \([2]\). The labeled and unlabeled variants are L-LIFT-RP and U-LIFT-RP, respectively. (We have only included this model for completeness.)