Polynomial Precise Interval Analysis Revisited

Thomas Gawlitza[1], Jérôme Leroux[2], Jan Reineke[3], Helmut Seidl[1], Grégoire Sutre[2], and Reinhard Wilhelm[3]

[1] TU München, Institut für Informatik, I2
80333 München, Germany
{gawlitza,seidl}@in.tum.de
[2] LaBRI, Université de Bordeaux, CNRS
33405 Talence Cedex, France
{leroux,sutre}@labri.fr
[3] Universität des Saarlandes, Germany
{reineke,wilhelm}@cs.uni-sb.de

Abstract. We consider a class of arithmetic equations over the complete lattice of integers (extended with $-\infty$ and ∞) and provide a polynomial time algorithm for computing least solutions. For systems of equations with addition and least upper bounds, this algorithm is a smooth generalization of the Bellman-Ford algorithm for computing the single source shortest path in presence of positive and negative edge weights. The method then is extended to deal with more general forms of operations as well as minima with constants. For the latter, a controlled widening is applied at loops where unbounded increase occurs. We apply this algorithm to construct a cubic time algorithm for the class of interval equations using least upper bounds, addition, intersection with constant intervals as well as multiplication.

1 Introduction

Interval analysis tries to derive tight bounds for the run-time values of variables [1]. This basic information may be used for important optimizations such as safe removals of array bound checks or for proofs of absence of overflows [2]. Since the very beginning of abstract interpretation, interval analysis has been considered as an algorithmic challenge. The reason is that the lattice of intervals may have infinite ascending chains. Hence, ordinary fixpoint iteration will not result in terminating analysis algorithms. The only general technique applicable here is the widening and narrowing approach of Cousot and Cousot [3]. If precision is vital, also more expressive domains are considered [4, 5]. While often returning amazingly good results, widening and narrowing typically does not compute the least solution of a system of equations but only a safe over-approximation.

In [6], however, Su and Wagner identify a class of interval equations for which the respective least solutions can be computed precisely and in polynomial time. As operations on intervals, they consider least upper bound, addition, scaling with positive and negative constants and intersection with constant intervals. The exposition of their algorithms, though, is not very explicit. Due to the

S. Albers, H. Alt, and S. Näher (Eds.): Festschrift Mehlhorn, LNCS 5760, pp. 422–437, 2009.

importance of the problem, we present an alternative and, hopefully more transparent approach. In particular, our methods also show how to deal with arbitrary multiplications of intervals. Our algorithm demonstrates how well-known ideas need only to be slightly extended to provide a both simple and efficient solution.

We start by investigating equations over integers only (extended with $-\infty$ and ∞ as least and greatest elements of the lattice) using maximum, addition, scaling with positive constants and minimum with constants as operations. In absence of minima, computing the least solution of such a system of equations can be considered as a generalization of the single-source shortest path problem from graphs to grammars in presence of positive and negative edge weights. A corresponding generalization for positive weights has been considered by Knuth [7]. Negative edge weights, though, complicate the problem considerably. While Knuth's algorithm can be considered as a generalization of Dijkstra's algorithm, we propose a generalization of the Bellman-Ford algorithm.

More generally, we observe that the Bellman-Ford algorithm works for all systems of equations which use operators satisfying a particular semantic property which we call BF-property. Beyond addition and multiplication with positive constants, positive as well as negative multiplication satisfies this property. *Positive* multiplication returns the product only if both arguments are positive, while *negative* multiplication returns the negated product if both arguments are negative. In order to obtain a polynomial algorithm also in presence of minima with constants, we instrument the basic Bellman-Ford algorithm to identify loops along which values might increase unboundedly. Once we have short-circuited the possibly costly iteration of such a loop we restart the Bellman-Ford algorithm until no further increments are found.

In the next step, we consider systems of equations over intervals using least upper bound, addition, negation, multiplication with positive constants as well as intersections with constant intervals and arbitrary multiplication of intervals. We show that computing the least solution of such systems can be reduced to computing the least solution of corresponding systems of integer equations. This reduction is inspired by the methods from [8] for interval equations with unrestricted intersections and the ideas of Leroux and Sutre [9], who first proved that interval equations with intersections with constant intervals as well as full multiplication can be solved in cubic time.

The rest of the paper is organized as follows. In Section 2, we introduce basic notions and consider methods for general systems of equations over $\mathcal{Z}$. Then we consider two classes of systems of equations over $\mathcal{Z}$ where least solutions can be computed in polynomial time. In Section 3, we consider systems of integer equations without minimum. In Section 4, we extend these methods to systems of equations where right-hand sides are *Bellman-Ford* functions. These systems can be solved in quadratic time (if arithmetic operations are executed in constant time). In Section 5, we then present our cubic time procedure for computing least solutions of systems of integer equations which additionally use minima with constants. In Section 6, we apply these techniques to construct a cubic