Corollary 8.11 of the previous chapter shows that there exists no robust linear approximation method for causal and stable transfer functions $f \in A(D)$ which are defined only on a finite set of discrete sampling points. It was discussed that these convergence problems of the approximation methods are a consequence of the fact that the approximation operator is only defined on a finite, discrete sampling set. However, the sampling of the given data is essential in practical applications, since nowadays numerical calculations are (almost) exclusively carried out on digital computers and such digital computers can process only a finite number of input data. For these reasons, the present chapter will discuss the consequences of the sampling of the given data for the behavior of certain numerical algorithms, a little bit more. Thereby, we will mainly focus on the calculation of the Hilbert transform from sampled data. However, these results carry over directly to algorithms for the calculation of the spectral factorization, Wiener filter or any other algorithm which involves explicitly or implicitly the determination of the algebraic conjugate of a given function.

Thus, we investigate algorithms which determine the Hilbert transform $\tilde{f} = Hf$ of a function $f$ given on the unit circle\(^1\). Since both $f$ and $\tilde{f}$ are defined only on $\mathbb{T}$, we will write $f(\theta)$ instead of $f(e^{i\theta})$ with $\theta \in [-\pi, \pi]$, throughout this chapter, to simplify the notations. Moreover, $\mathbb{T}$ will now stand for the interval $[-\pi, \pi)$ of the real axis $\mathbb{R}$ and $C(\mathbb{T})$ denotes the set of all continuous functions $f$ on $\mathbb{T}$ with $f(-\pi) = f(\pi)$ or equivalently for all continuous, $2\pi$-periodic functions on $\mathbb{R}$.

We consider linear operators $\mathcal{F}$ which determine an approximation of the Hilbert transform $\tilde{f} = Hf$ from the values $f(\tau_k)$ of the given function $f$ on a finite set $S = \{\tau_k \in \mathbb{T} : k = 1, 2, \cdots, N\}$ of sampling points $\tau_k$, only. Denote by $\mathcal{F}_N$ such a linear operator which approximates the Hilbert transform based on a sampling set $S$ of cardinality $N$. Then we say that a sequence $\{\mathcal{F}_N\}_{N \in \mathbb{N}}$ of such operators approximates the conjugate function $\tilde{f}$ of $f \in C(\mathbb{T})$ arbitrarily.

\(^1\) See Section 5.3 for the definition of the Hilbert transform.

well (in the norm of $C(T)$), if

$$\lim_{N \to \infty} \|\tilde{f} - \Xi_N f\|_\infty = 0.$$ 

In the following it is assumed that the operators $\Xi$ have the following two natural properties.

**Definition 9.1 (Property I).** We say that an operator $\Xi$ has the property I if it is linear, i.e.

$$\Xi(f_1 + f_2) = \Xi f_1 + \Xi f_2 \quad \text{and} \quad \Xi(\lambda f) = \lambda \Xi(f)$$

for all $\lambda \in \mathbb{C}$, and if $\Xi$ is concentrated on the set $S$ of sampling points, i.e. if two functions $f_1$ and $f_2$ coincide on the sampling point $f_1(\tau_k) = f_2(\tau_k)$ for all $\tau_k \in S$ then $(\Xi f_1)(t) = (\Xi f_2)(t)$ for all $t \in T$.

This property requires the linearity of the operator $\Xi$ and the concentration of the operator on the finite set $S$, i.e. if two functions $f_1$ and $f_2$ coincide on the sampling set $S$, the operator $\Xi$ will give the same result for both functions. It is immediately clear that all practical algorithms for the calculation of the conjugate function $\tilde{f}$ from $f$ which can be implemented on a digital computer, have to satisfy this property since generally only a finite number of values can be taken into account during the calculation on such a computer. Therefore, this property is practically no limitation on the linear operators under consideration.

**Example 9.2.** The conjugate Shannon sampling series, given by

$$(\Xi_N f)(t) = \frac{1}{2N + 1} \sum_{k=0}^{2N} f\left(\frac{2\pi k}{2N+1}\right) \cos \frac{2N+1}{2} \left(t - \frac{2\pi k}{2N+1}\right) - \cos \frac{1}{2} \left(t - \frac{2\pi k}{2N+1}\right) \sin \frac{1}{2} \left(t - \frac{2\pi k}{2N+1}\right)$$

is one example of such an operator. It calculates an approximation of the conjugate function $\tilde{f}$ based on the function $f$ given only at the points of the sampling set $S = \{\tau_k = 2\pi k/(2N + 1) : k = 0, 1, \ldots, 2N\}$.

Example 5.9 shows that there exist continuous functions $f \in C(T)$ such that the conjugate $\tilde{f} = \hat{y} f$ is not continuous on $T$. For this reason, it cannot be expected that a linear operator $\Xi$ approximates the Hilbert transform of such functions arbitrary well, in general. Therefore the operators $\Xi$ are considered only on the set $B$ of all continuous functions which have a continuous Hilbert transform:

$$B := \{ f \in C(T) : \tilde{f} = \hat{y} f \in C(T) \}$$

The norm in $B$ is defined by $\|f\|_B := \max(\|f\|_\infty, \|\tilde{f}\|_\infty)$. Thus, the Hilbert transform $\hat{y}$ maps every function $f \in B$ onto the continuous function $\tilde{f} = \hat{y} f$ which shows that $\hat{y} : B \to C(T)$ is a continuous mapping with $\|\hat{y}\|_{B \to C(T)} \leq 1$. Note, that if a function $f \in B$ is approximated by a trigonometric polynomial