The theory of solutions of genuinely nonlinear, strictly hyperbolic systems of two conservation laws will be developed in this chapter at a level of precision comparable to that for genuinely nonlinear scalar conservation laws, expounded in Chapter XI. This will be achieved by exploiting the presence of coordinate systems of Riemann invariants and the induced rich family of entropy-entropy flux pairs. The principal tools in the investigation will be generalized characteristics and entropy estimates.

The analysis will reveal a close similarity in the structure of solutions of scalar conservation laws and pairs of conservation laws. Thus, as in the scalar case, jump discontinuities are generally generated by the collision of shocks and/or the focusing of compression waves, and are then resolved into wave fans approximated locally by the solution of associated Riemann problems.

The total variation of the trace of solutions along space-like curves is controlled by the total variation of the initial data, and spreading of rarefaction waves affects total variation, as in the scalar case.

The dissipative mechanisms encountered in the scalar case are at work here as well, and have similar effects on the large-time behavior of solutions. Entropy dissipation induces $O(t^{-1/2})$ decay of solutions with initial data in $L^1(-\infty, \infty)$. When the initial data have compact support, the two characteristic families asymptotically decouple; the characteristics spread and form a single $N$-wave profile for each family. Finally, as in the scalar case, confinement of characteristics under periodic initial data induces $O(t^{-1})$ decay in the total variation per period and formation of sawtoothed profiles, one for each characteristic family.

12.1 Notation and Assumptions

We consider a genuinely nonlinear, strictly hyperbolic system of two conservation laws,

\begin{equation}
\partial_t U(x,t) + \partial_x F(U(x,t)) = 0,
\end{equation}

on some disk $\mathcal{O}$ centered at the origin. The eigenvalues of $DF$ (characteristic speeds) will here be denoted by $\lambda$ and $\mu$, with $\lambda(U) < 0 < \mu(U)$ for $U \in \mathcal{O}$, and the associated eigenvectors will be denoted by $R$ and $S$.

The system is endowed with a coordinate system $(z, w)$ of Riemann invariants, vanishing at the origin $U = 0$, and normalized according to (7.3.8):

\begin{align*}
DzR &= 1, \quad DzS = 0, \quad DwR = 0, \quad DwS = 1.
\end{align*}

The condition of genuine nonlinearity is now expressed by (7.5.4), which here reads

\begin{align*}
\lambda_z < 0, \quad \mu_w > 0.
\end{align*}

The direction in the inequalities (12.1.3) has been selected so that $z$ increases across admissible weak 1-shocks while $w$ decreases across admissible weak 2-shocks.

For definiteness, we will consider systems with the property that the interaction of any two shocks of the same characteristic family produces a shock of the same family and a rarefaction wave of the opposite family. Note that this condition is here expressed by

\begin{align*}
S^\top D^2zS > 0, \quad R^\top D^2wR > 0.
\end{align*}

Indeed, in conjunction with (8.2.24), (12.1.3) and Theorem 8.3.1, the inequalities (12.1.4) imply that $z$ increases across admissible weak 2-shocks while $w$ decreases across admissible weak 1-shocks. Therefore, the admissible shock and rarefaction wave curves emanating from the state $(\bar{z}, \bar{w})$ have the shape depicted in Fig. 12.1.1. Consequently, as seen in Fig. 12.1.2(a), a 2-shock that joins the state $(z_\ell, w_\ell)$, on the left, with the state $(z_m, w_m)$, on the right, interacts with a 2-shock that joins $(z_m, w_m)$, on the left, with the state $(z_r, w_r)$, on the right, to produce a 1-rarefaction wave, joining $(z_\ell, w_\ell)$, on the left, with a state $(z_0, w_\ell)$, on the right, and a 2-shock joining $(z_0, w_\ell)$, on the left, with $(z_r, w_r)$, on the right, as depicted in Fig. 12.1.2(b). Similarly, the interaction of two 1-shocks produces a 1-shock and a 2-rarefaction wave.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig12.1.1.png}
\caption{Fig. 12.1.1}
\end{figure}